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ON GENERALIZED ANALYSIS OF VARIANCE. (I)

BY P. L. HSU

1. *The Wilks-Lawley hypothesis.* Suppose that, as a result of some random sampling, we are in possession of $p(n_1+n)$ quantities, where $n \geqslant p$, calculated from the observational data. Calling these y_{ir} and $z_{ir'}$ $(i=1,2,\ldots,p;\ r=1,2,\ldots,n_1;\ r'=1,2,\ldots,n)$, we assume that in repeated sampling they follow the probability distribution

$$\text{Const.} \times \exp\left\{-\frac{1}{2}\sum_{i,j=1}^{p}\alpha_{ij}\sum_{r=1}^{n_1}(y_{ir}-\eta_{ir})(y_{jr}-\eta_{jr})-\frac{1}{2}\sum_{i,j=1}^{p}\alpha_{ij}\sum_{r=1}^{n}z_{ir}z_{jr}\right\}\Pi dydz, \quad (1)$$

and that we have no previous knowledge of the values of the η_{ir}. Our problem is to test the hypothesis H_0:

$$\eta_{ir}=0 \quad \text{for} \quad i=1,2,\ldots,p;\ r=1,2,\ldots,n_1. \qquad (H_0)$$

For $p=1$ the hypothesis is that to which every linear hypothesis* may be reduced, and the test amounts to that ordinarily employed in the analysis of variance. For $n_1=1$ the problem calls for Hotelling's generalized "Student's" test.†

It should be emphasized that the y_{ir}, z_{ir} and η_{ir} will not usually correspond to the original physical observations and physical constants belonging to the sampled populations, but are so derived from them that (1) is true and that H_0 is equivalent to some hypothesis regarding population constants that we want to test. In another paper we shall deal with a general class of "linear hypothesis" on multivariate normal means, showing how all of them can be reduced by a rotation of the sample space to the "canonical" H_0. Here we content ourselves with the following example.

Example. Case of k samples. x_{ivt} $(i=1,2,\ldots,p;\ v=1,2,\ldots,k;\ t=1,2,\ldots,m_v)$ are k samples drawn respectively from k p-variate normal populations all of which have the same set of variances and covariances. Let the population means be ξ_{iv} $(i=1,2,\ldots,p;\ v=1,2,\ldots,k)$. Let

$$\sum_{v=1}^{k}m_v=M, \quad \bar{\xi}_i=\frac{1}{M}\sum_{v=1}^{k}m_v\xi_{iv},$$

$$\bar{x}_{iv}=\frac{1}{m_v}\sum_{t=1}^{m_v}x_{ivt}, \quad \bar{x}_i=\frac{1}{M}\sum_{v=1}^{k}m_v\bar{x}_{iv},$$

$$s_{ijv}=\sum_{t=1}^{m_v}(x_{ivt}-\bar{x}_{iv})(x_{jvt}-\bar{x}_{jv}),$$

$$(i,j=1,2,\ldots,p;\ v=1,2,\ldots,k).$$

* Kolodziejczyk (1935), Tang (1938).

† Hotelling (1931), Hsu (1938), Bose and Roy (1938)

Suppose that we want to test the hypothesis:

$$\xi_{i1} = \xi_{i2} = \ldots = \xi_{ik} \quad \text{for} \quad i = 1, 2, \ldots, p.$$

Then it is known that if we call y_{ir} and z_{ir} certain linear functions of the x_{iv} and η_{ir} certain linear functions of the ξ_{iv}, (1) will hold true and the hypothesis under test is equivalent to H_0. Thus reducing the problem, we have

$$n_1 = k-1, \quad n = M-k,$$

$$\sum_{r=1}^{n_1} y_{ir}y_{jr} = \sum_{\nu=1}^{k} m_\nu(\bar{x}_{i\nu}-\bar{x}_i)(\bar{x}_{j\nu}-\bar{x}_j), \quad \sum_{r=1}^{n} z_{ir}z_{jr} = \sum_{\nu=1}^{k} s_{ij\nu},$$

$$\sum_{r=1}^{n_1} \eta_{ir}\eta_{jr} = \sum_{\nu=1}^{k} m_\nu(\xi_{i\nu}-\bar{\xi}_i)(\xi_{j\nu}-\bar{\xi}_j),$$

$$(i, j = 1, 2, \ldots, p).$$

In particular, if, $k = 2$ then $n_1 = 1$, and we may drop the second index of y and η. We have then

$$y_i y_j = \frac{m_1 m_2}{m_1 + m_2}(\bar{x}_{i1}-\bar{x}_{i2})(\bar{x}_{j1}-\bar{x}_{j2}),$$

$$\sum_{r=1}^{n} z_{ir}z_{jr} = \sum_{t=1}^{m_1}(x_{it}-\bar{x}_{i1})(x_{jt}-\bar{x}_{j1}) + \sum_{t=1}^{m_2}(x_{it}-\bar{x}_{i2})(x_{jt}-\bar{x}_{j2}),$$

$$\eta_i \eta_j = \frac{m_1 m_2}{m_1 + m_2}(\xi_{i1}-\xi_{i2})(\xi_{j1}-\xi_{j2}),$$

$$(i, j = 1, 2, \ldots, p).$$

The hypothesis in the above example has been studied by Wilks (1932), while an attempt to test the general hypothesis H_0 was made by Lawley (1938). Both assumed no prior knowledge of the values of the α_{ij}. We propose to call H_0 the Wilks-Lawley hypothesis.

2. *Case where the α_{ij} are known, generalizations of Mahalanobis's distance.* From now on we shall write

$$a_{ij} = \sum_{r=1}^{n_1} y_{ir}y_{jr}, \quad b_{ij} = \sum_{r=1}^{n} z_{ir}z_{jr},$$

$$\psi_{ij} = \sum_{r=1}^{n_1} \eta_{ir}\eta_{jr}, \quad \Psi = \sum_{i,j=1}^{p} \alpha_{ij}\psi_{ij},$$

$$(i, j = 1, 2, \ldots, p).$$

In order to test H_0 when the α_{ij} are known, we calculate the likelihood ratio* and get

$$-2\log(\text{likelihood ratio}) = \sum_{i,j=1}^{p} \alpha_{ij}a_{ij} = S, \text{ say}$$

and decide to reject H_0 if the value of S exceeds some fixed constant, chosen so as to fix at some desired level the risk of rejecting H_0 when it is true.

* For the definition and interpretation of the term see Neyman & Pearson (1928).

THEOREM 1. *For arbitrary values of the η_{ir} the distribution of S is that of the sum of pn_1 non-central squares:**

$$2^{-\frac{1}{2}pn_1}S^{\frac{1}{2}pn_1-1}e^{-\frac{1}{2}(S+\Psi)}\left\{\sum_{h=0}^{\infty}\frac{\Psi^h S^h}{4^h h!\Gamma(h+\frac{1}{2}pn_1)}\right\}dS. \tag{2}$$

In particular, if H_0 is true, then $\Psi = 0$ and S follows the χ^2 distribution with pn_1 degrees of freedom.

Proof. Let the sets of variables $(y_{1r}, y_{2r}, \ldots, y_{pr})$, for $r = 1, 2, \ldots, n_1$, be subject to the same linear transformation such that, calling the new variables $(u_{1r}, u_{2r}, \ldots, u_{pr})$,

$$\sum_{i,j=1}^{p}\alpha_{ij}y_{ir}y_{jr} = \sum_{i=1}^{p}u_{ir}^2, \quad (r = 1, 2, \ldots, n_1).$$

This is possible because the matrix $\|\alpha_{ij}\|$ is positive definite. Let μ_{ir} be the same linear function of the η's as u_{ir} is of the y's. Then

$$S = \sum_{i=1}^{p}\sum_{r=1}^{n_1}u_{ir}^2 \tag{3}$$

and the u_{ir} follow the distribution

$$(2\pi)^{-\frac{1}{2}pn_1}\exp\left\{-\frac{1}{2}\sum_{i=1}^{p}\sum_{r=1}^{n_1}(u_{ir}-\mu_{ir})^2\right\}\Pi du. \tag{4}$$

From (3) and (4) follows the result (2),† as we have

$$\sum_{i=1}^{p}\sum_{r=1}^{n_1}\mu_{ir}^2 = \sum_{i,j=1}^{p}\alpha_{ij}\sum_{r=1}^{n_1}\eta_{ir}\eta_{jr} = \Psi.$$

From (2) we get
$$\mathscr{E}(S) = \Psi + pn_1, \tag{5}$$

$$\mathscr{E}(S^2) = \Psi^2 + 2(pn_1+2)\Psi + pn_1(pn_1+2),$$

whence
$$\sigma^2(S) = 4\Psi + 2pn_1.$$

If these general results are applied to the above example of k samples, we have

$$S = \sum_{i,j=1}^{p}\alpha_{ij}\sum_{\nu=1}^{k}m_\nu(\bar{x}_{i\nu}-\bar{x}_i)(\bar{x}_{j\nu}-\bar{x}_j) = S_k,$$

$$\Psi = \sum_{i,j=1}^{p}\alpha_{ij}\sum_{\nu=1}^{k}m_\nu(\xi_{i\nu}-\bar{\xi}_i)(\xi_{j\nu}-\bar{\xi}_j) = \Psi_k,$$

say, and the distribution

$$2^{-\frac{1}{2}p(k-1)}S_k^{\frac{1}{2}p(k-1)-1}e^{-\frac{1}{2}(S_k+\Psi_k)}\left\{\sum_{h=0}^{\infty}\frac{\Psi_k^h S_k^h}{4^h h!\Gamma(h+\frac{1}{2}p(k-1))}\right\}dS_k. \tag{6}$$

For $k = 2$ we have

$$S_2 = \frac{m_1m_2}{m_1+m_2}\sum_{i,j=1}^{p}\alpha_{ij}(\bar{x}_{i1}-\bar{x}_{i2})(\bar{x}_{j1}-\bar{x}_{j2}),$$

$$\Psi_2 = \frac{m_1m_2}{m_1+m_2}\sum_{i,j=1}^{p}\alpha_{ij}(\xi_{i1}-\xi_{i2})(\xi_{j1}-\xi_{j2}),$$

* Fisher (1928), p. 669. † Fisher (1928), Tang (1938), pp. 138–9.

and the distribution

$$2^{-\frac{1}{2}p}S_2^{\frac{1}{2}p-1}e^{-\frac{1}{2}(S_2+\Psi_2)}\left(\sum_{h=0}^{\infty}\frac{\Psi_2^h S_2^h}{4^h h!\Gamma(h+\frac{1}{2}p)}\right)dS_2. \tag{7}$$

The quantity Ψ_2 is equal to

$$\frac{pm_1m_2}{m_1+m_2}\Delta,$$

where Δ is Mahalanobis's "distance"* between two populations having the same variances and covariances weighted according to m_1 and m_2. The statistic

$$\frac{m_1+m_2}{pm_1m_2}S_2-\frac{1}{m_1}-\frac{1}{m_2},$$

which, according to (5), is an unbiassed estimate of Δ, is called the D^2 statistic and the distribution (7) has already been obtained.† These concepts and formulae are completely generalized here by S_k, Ψ_k and the formulae (5) and (6) for the case $k>2$.

3. *Tests suggested by Wilks and Lawley: Generalized E^2, $1-E^2$ and $E^2(1-E^2)^{-1}$.* From now on we shall assume complete ignorance of the values of the α_{ij}. In the case $p=1$ the following functions are well known:

$$E^2=\sum_{r=1}^{n_1}y_r^2\Big/\left(\sum_{r=1}^{n_1}y_r^2+\sum_{r=1}^{n}z_r^2\right),$$

$$1-E^2=\sum_{r=1}^{n}z_r^2\Big/\left(\sum_{r=1}^{n_1}y_r^2+\sum_{r=1}^{n}z_r^2\right),$$

$$E^2/(1-E^2)=\sum_{r=1}^{n_1}y_r^2\Big/\sum_{r=1}^{n}z_r^2.$$

Wilks, while studying the k-sample case,‡ suggested the functions U and W respectively as generalizations of E^2 and $1-E^2$. In our general case these are defined as

$$U=\frac{|a_{ij}|}{|a_{ij}+b_{ij}|}\quad\text{if } n_1\geqslant p, \tag{8}$$

and

$$W=\frac{|b_{ij}|}{|a_{ij}+b_{ij}|}. \tag{9}$$

Both are ratios of two determinants. The above definition for U based on the idea of a generalized E^2 breaks down if $n_1<p$, as then $|a_{ij}|$ vanishes identically. This difficulty can be met if we define U as the product of the non-identically vanishing roots of the determinantal equation $|a_{ij}-\theta(a_{ij}+b_{ij})|=0$. If $n_1\geqslant p$, this definition evidently coincides with (8).

* Mahalanobis (1936). † Bose (1936). ‡ Wilks (1932).

As a generalization of $E^2(1-E^2)^{-1}$ we take, instead of UW^{-1}, the function V suggested by Lawley* and defined as

$$V = \sum_{i,j=1}^{p} b^{ij} a_{ij}, \tag{10}$$

where b^{ij} is the element (i, j) of the reciprocal matrix $\| b_{ij} \|^{-1}$.

If we denote by $\theta_1, \theta_2, \ldots, \theta_{l_1}$, where l_1 is the smaller of the integers p and n_1, the non-identically vanishing roots of the equation

$$| a_{ij} - \theta(a_{ij} + b_{ij}) | = 0,$$

then, according to (9) and (10) and the new definition of V, we have

$$U = \prod_{i=1}^{l_1} \theta_i, \tag{11}$$

$$W = \prod_{i=1}^{l_1} (1-\theta_i), \tag{12}$$

$$V = \sum_{i=1}^{l_1} \frac{\theta_i}{1-\theta_i}. \tag{13}$$

W and V are test functions for H_0 suggested respectively by Wilks and Lawley. If we use W (or V), we reject H_0 if W is smaller (or V is greater) than some prescribed constant. In the following sections we shall study the distributions of U, V and W in repeated sampling.

4. *Case when H_0 is true.* If H_0 is true, then, putting all the $\eta_{ir} = 0$ in (1), we get the parent distribution

$$\text{const.} \times \exp\left\{ -\frac{1}{2} \sum_{i,j=1}^{p} \alpha_{ij}(a_{ij} + b_{ij}) \right\} \Pi \, dy \, dz. \tag{14}$$

The following theorem has been proved elsewhere.†

THEOREM 2. *Let l_1 be the smaller, and l_2 the larger, of the integers p and n_1. Let $\theta_1, \theta_2, \ldots, \theta_{l_1}$ be the non-identically vanishing roots of the determinantal equation*

$$| a_{ij} - \theta(a_{ij} + b_{ij}) | = 0, \tag{15}$$

arranged in the order of descending magnitude: $1 \geqslant \theta_1 \geqslant \theta_2 \geqslant \ldots \geqslant \theta_{l_1} \geqslant 0$. Then the simultaneous distribution of the θ's, as derived from (14), *is*

$$\pi^{\frac{1}{2}l_1} \left\{ \prod_{i=1}^{l_1} \frac{\Gamma\frac{1}{2}(n-p+l_2+i)}{\Gamma\frac{1}{2}(l_2-l_1+i)\,\Gamma\frac{1}{2}(n-p+i)\,\Gamma\frac{1}{2}i} \right\} \times \left\{ \prod_{i=1}^{l_1} \theta_i \right\}^{\frac{1}{2}(l_2-l_1-1)} \left\{ \prod_{i=1}^{l_1} (1-\theta_i) \right\}^{\frac{1}{2}(n-p-1)} \left\{ \prod_{i=1}^{l_1} \prod_{j=i+1}^{l_1} (\theta_i - \theta_j) \right\} \left\{ \prod_{i=1}^{l_1} d\theta_i \right\}. \tag{16}$$

Therefore the distributions of U, W and V are those of the functions in (11), (12) and (13), where the θ's follow the distribution (16).

* Lawley (1938). † Hsu (1939), Fisher (1939).

Take the particular case $l_1 = 2$. We have then

$$U = \theta_1\theta_2, \quad W = (1-\theta_1)(1-\theta_2), \quad V = \frac{\theta_1}{1-\theta_1} + \frac{\theta_2}{1-\theta_2},$$

whence $$U^{\frac{1}{2}} + W^{\frac{1}{2}} \leqslant 1, \quad V = W^{-1}(1-U-W),$$

and the distribution (16) becomes

$$\frac{\Gamma(n-p+l_2+1)}{4\Gamma(l_2-1)\Gamma(n-p+1)} (\theta_1\theta_2)^{\frac{1}{2}(l_2-3)} \{(1-\theta_1)(1-\theta_2)\}^{\frac{1}{2}(n-p-1)} (\theta_1-\theta_2)\, d\theta_1 d\theta_2. \quad (17)$$

From (17) we easily obtain the joint distribution of U and W:

$$\frac{\Gamma(n-p+l_2+1)}{4\Gamma(l_2-1)\Gamma(n-p+1)} U^{\frac{1}{2}(l_2-3)} W^{\frac{1}{2}(n-p-1)} dU\, dW.$$

Integrating with respect to W ranging from 0 to $(1-U^{\frac{1}{2}})^2$, we get the distribution of U:

$$\frac{1}{2B(l_2-1, n-p+2)} U^{\frac{1}{2}(l_2-3)} (1-U^{\frac{1}{2}})^{n-p+1} dU. \quad (18)$$

Similarly, the distribution of W is

$$\frac{1}{2B(l_2, n-p+1)} W^{\frac{1}{2}(n-p-1)} (1-W^{\frac{1}{2}})^{l_2-1} dW. \quad (19)$$

Again, from (18) we find the distribution of V:

$$\frac{\Gamma(n-p+l_2+1)}{4\Gamma(l_2-1)\Gamma(n-p+1)} (1+V)^{-\frac{1}{2}(n-p+3)} \left\{ \int_{\frac{4(1+V)}{(2+V)^2}}^{1} y^{\frac{1}{2}(n-p+1)} (1-y)^{\frac{1}{2}(l_2-3)} dy \right\} dV. \quad (20)$$

Now in the particular case considered where $l_1 = 2$ we have either (i) $p = 2$, $l_2 = n_1$ if $p \leqslant n_1$, or (ii) $n_1 = 2$, $l_2 = p$ if $n_1 \leqslant p$. Hence, from (18), (19) and (20) we get the following three pairs of distributions:

$$\frac{1}{2B(n_1-1, n)} U^{\frac{1}{2}(n_1-3)} (1-U^{\frac{1}{2}})^{n-1} dU, \quad 2 = p \leqslant n_1,$$

$$\frac{1}{2B(p-1, n-p+2)} U^{\frac{1}{2}(p-3)} (1-U^{\frac{1}{2}})^{n-p+1} dU, \quad 2 = n_1 \leqslant p,$$

$$\frac{1}{2B(n_1, n-1)} W^{\frac{1}{2}(n-3)} (1-W^{\frac{1}{2}})^{n_1-1} dW, \quad 2 = p \leqslant n_1,$$

$$\frac{1}{2B(p, n-p+1)} W^{\frac{1}{2}(n-p-1)} (1-W^{\frac{1}{2}})^{p-1} dW, \quad 2 = n_1 \leqslant p,$$

$$\frac{\Gamma(n_1+n-1)}{4\Gamma(n_1-1)\Gamma(n-1)} (1+V)^{-\frac{1}{2}(n+1)} dV \int_{4(1+V)(2+V)^{-2}}^{1} y^{\frac{1}{2}(n-1)} (1-y)^{\frac{1}{2}(n_1-3)} dy, \quad 2 = p \leqslant n_1,$$

$$\frac{\Gamma(n+1)}{4\Gamma(p-1)\Gamma(n-p+1)} (1+V)^{-\frac{1}{2}(n-p+3)} dV \int_{4(1+V)(2+V)^{-2}}^{1} y^{\frac{1}{2}(n-p+1)} (1-y)^{\frac{1}{2}(p-3)} dy, \quad 2 = n_1 \leqslant p.$$

The general expression for the moments of U and W can easily be deduced from (16). We have, from (16),

$$C(l_2, n)\int f(\theta; l_2, n)\prod_{i=1}^{l_1} d\theta_i = 1,$$

where
$$C(l_2, n) = \pi^{\frac{1}{2}l_1}\prod_{i=1}^{l_1}\frac{\Gamma\frac{1}{2}(n-p+l_2+i)}{\Gamma\frac{1}{2}(l_2-l_1+i)\,\Gamma\frac{1}{2}(n-p+i)\,\Gamma\frac{1}{2}i},$$

$$f(\theta; l_2, n) = \left\{\prod_{i=1}^{l_1}\theta_i\right\}^{\frac{1}{2}(l_2-l_1-1)}\left\{\prod_{i=1}^{l_1}(1-\theta_i)\right\}^{\frac{1}{2}(n-p-1)}\left\{\prod_{i=1}^{l_1}\prod_{j=i+1}^{l_1}(\theta_i-\theta_j)\right\}.$$

Hence
$$\int f(\theta; l_2, n)\prod_{i=1}^{l_1} d\theta_i = \frac{1}{C(l_2, n)},$$

whence
$$\mathscr{E}(U^{q_1}W^{q_2}) = C(l_2, n)\int f(\theta; l_2+q_1, n+q_2)\prod_{i=1}^{l_1} d\theta_i = \frac{C(l_2, n)}{C(l_2+2q_1, n+2q_2)}$$

$$= \prod_{i=1}^{l_1}\frac{\Gamma\frac{1}{2}(n-p+l_2+i)\,\Gamma\frac{1}{2}(l_2-l_1+2q_1+i)\,\Gamma\frac{1}{2}(n+2q_2-p+i)}{\Gamma\frac{1}{2}(n-p+l_2+2q_1+2q_2+i)\,\Gamma\frac{1}{2}(l_2-l_1+i)\,\Gamma\frac{1}{2}(n-p+i)}.$$

The case of k samples (see p. 221) of two variables ($p=2$) has been studied at length by Pearson & Wilks (1933). The hypothesis H_0 and the functions U and W are called in their paper H_2, U_2 and L_2 respectively. It is found there that the test based on U_2 (i.e. U) cannot be regarded as an adequate test of H_0. We shall show in the next section that H_0 is true if and only if two population constants, called λ_1 and λ_2, both vanish. The disadvantage of U referred to by Pearson and Wilks, may be expressed by saying that it is unlikely to be able to detect the falsehood of H_0 if only one of λ_1 and λ_2 vanishes.

5. *Simplification of the parent distribution function.* The matrix $\|\alpha_{ij}\|$ is positive definite; hence it can be expressed as $\mathbf{CC}'$,* where $\mathbf{C}$ is some non-singular real matrix. Write

$$\mathbf{Y} = \begin{Vmatrix} y_{11} & y_{12} & \cdots & y_{1n_1} \\ y_{21} & y_{22} & \cdots & y_{2n_1} \\ \cdots & \cdots & \cdots & \cdots \\ y_{p1} & y_{p2} & \cdots & y_{pn_1} \end{Vmatrix}, \quad \mathbf{Z} = \begin{Vmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ z_{p1} & z_{p2} & \cdots & z_{pn} \end{Vmatrix}, \quad \mathbf{G} = \begin{Vmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1n_1} \\ \eta_{21} & \eta_{22} & \cdots & \eta_{2n_1} \\ \cdots & \cdots & \cdots & \cdots \\ \eta_{p1} & \eta_{p2} & \cdots & \eta_{pn_1} \end{Vmatrix},$$

$$\|\sigma_{ij}\| = \|\alpha_{ij}\|^{-1} = (\mathbf{C}')^{-1}\mathbf{C}^{-1}.$$

THEOREM 3. *Let* $\mathbf{G}$ *be of rank* l *(hence* $l \leqslant p$, $l \leqslant n_1$ *and* l *is the rank of* $GG' = \|\psi_{ij}\|$*), and let* $\lambda_1, \lambda_2, \ldots, \lambda_l$ *be the non-vanishing roots of the determinantal equation* $|\psi_{ij} - \lambda\sigma_{ij}| = 0$. *Let* $\theta_1, \theta_2, \ldots, \theta_{l_1}$ *be the non-identically vanishing roots of the determinantal equation* $|a_{ij} - \theta(a_{ij}+b_{ij})| = 0$. *Then the joint distribution of the* θ*'s, as derived from* (1)*, depends upon the parameters* $\lambda_1, \lambda_2, \ldots, \lambda_l$ *alone in such a way that instead of* (1) *we may regard the parent distribution as*

$$(2\pi)^{-\frac{1}{2}p(n_1+n)}\exp\left(-\tfrac{1}{2}\Psi\right)\exp\left\{-\frac{1}{2}\sum_{i=1}^{p}a_{ii} - \frac{1}{2}\sum_{i=1}^{p}b_{ii} + \sum_{i=1}^{l}\sqrt{\lambda_i}\,y_{ii}\right\}\Pi dy\,dz. \quad (21)$$

* The accent denotes the transposed matrix.

Hence, in studying the distribution of any functions of the θ's, such as U, V and W, we may replace (1) by (21).

LEMMA 1 (Hotelling).* *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be any positive definite matrices of orders* m *and* n *respectively, and* $\mathbf{C}$ *be any real matrix of order* $m \times n$ *and rank* l. *There exist two non-singular real matrices,* $\mathbf{N}_1$ *and* $\mathbf{N}_2$, *such that*

$$\mathbf{N}_1\mathbf{A}\mathbf{N}_1' = \mathbf{I},\dagger \quad \mathbf{N}_2\mathbf{B}\mathbf{N}_2' = \mathbf{I}, \quad \mathbf{N}_1\mathbf{C}\mathbf{N}_2' = \mathbf{D},$$

where $$\mathbf{D} = \begin{Vmatrix} \mathbf{D}_{\sqrt{\lambda}} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{Vmatrix},\dagger \qquad \mathbf{D}_{\sqrt{\lambda}} = \begin{Vmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \cdots & \\ & & & \sqrt{\lambda_l} \end{Vmatrix},$$

and the λ_i *are the non-vanishing roots of the determinantal equation*

$$|\mathbf{C}\mathbf{B}^{-1}\mathbf{C}' - \lambda\mathbf{A}| = 0.$$

Allowing both $\mathbf{A}$ and $\mathbf{B}$ to be unit matrices in Lemma 1 we get the following:

Corollary. Let $\mathbf{C}$ *be any real matrix of rank* l. *There exist two real orthogonal matrices* $\mathbf{\Gamma}_1$ *and* $\mathbf{\Gamma}_2$ *such that*

$$\mathbf{\Gamma}_1\mathbf{C}\mathbf{\Gamma}_2' = \mathbf{D}$$

where $$\mathbf{D} = \begin{Vmatrix} \mathbf{D}_{\sqrt{\lambda}} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{Vmatrix}, \qquad \mathbf{D}_{\sqrt{\lambda}} = \begin{Vmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \cdots & \\ & & & \sqrt{\lambda_l} \end{Vmatrix},$$

and the λ_i *are the non-vanishing latent roots of the matrix* $\mathbf{C}\mathbf{C}'$.

Proof of Theorem 3. We write $\operatorname{tr}\mathbf{A}$ for the sum of the diagonal elements of any square matrix $\mathbf{A}$. It is easily verified that $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$ whenever $\mathbf{AB}$ is a square matrix.

The expression inside the bracket in (1) may be written as

$$\begin{aligned} &-\tfrac{1}{2}\Psi - \tfrac{1}{2}\operatorname{tr}(\mathbf{C}\mathbf{C}'\mathbf{Y}\mathbf{Y}') - \tfrac{1}{2}\operatorname{tr}(\mathbf{C}\mathbf{C}'\mathbf{Z}\mathbf{Z}') + \operatorname{tr}(\mathbf{C}\mathbf{C}'\mathbf{G}\mathbf{Y}') \\ &\qquad = -\tfrac{1}{2}\Psi - \tfrac{1}{2}\operatorname{tr}(\mathbf{C}'\mathbf{Y}\mathbf{Y}'\mathbf{C}) - \tfrac{1}{2}\operatorname{tr}(\mathbf{C}'\mathbf{Z}\mathbf{Z}'\mathbf{C}) + \operatorname{tr}(\mathbf{C}'\mathbf{G}\mathbf{Y}'\mathbf{C}). \quad (22) \end{aligned}$$

The λ_i, as defined in Theorem 3, are the non-vanishing latent roots of the equation $|\mathbf{G}\mathbf{G}' - \lambda(\mathbf{C}')^{-1}\mathbf{C}^{-1}| = 0$. On pre- and post-multiplying by $|\mathbf{C}'|$ and $|\mathbf{C}|$ respectively this becomes $|\mathbf{C}'\mathbf{G}\mathbf{G}'\mathbf{C} - \lambda\mathbf{I}| = 0$. Hence the λ_i are the non-vanishing latent roots of the matrix $(\mathbf{C}'\mathbf{G})(\mathbf{C}'\mathbf{G})'$. Therefore, by the corollary to Lemma 1, there exist two real orthogonal matrices, $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$, such that

$$\mathbf{\Gamma}_1\mathbf{C}'\mathbf{G}\mathbf{\Gamma}_2 = \mathbf{D} = \begin{Vmatrix} \mathbf{D}_{\sqrt{\lambda}} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{Vmatrix}. \quad (23)$$

* Hotelling (1936), pp. 326–30.

† $\mathbf{I}$ and $\mathbf{O}$ stand for a unit matrix and a zero matrix respectively.

The transformation of variables

$$\mathbf{Y} = (\mathbf{C}')^{-1}\boldsymbol{\Gamma}_1'\mathbf{U}\boldsymbol{\Gamma}_2', \quad \mathbf{Z} = (\mathbf{C}')^{-1}\boldsymbol{\Gamma}_1'\mathbf{V}, \tag{24}$$

where $\mathbf{U} = \begin{Vmatrix} u_{11} & u_{12} & \cdots & u_{1n_1} \\ u_{21} & u_{22} & \cdots & u_{2n_1} \\ \cdots & \cdots & \cdots & \cdots \\ u_{p1} & u_{p2} & \cdots & u_{pn_1} \end{Vmatrix}$, $\mathbf{V} = \begin{Vmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ v_{p1} & v_{p2} & \cdots & v_{pn} \end{Vmatrix}$

are the matrices of the new variables, has a constant Jacobian and leaves the θ's invariant, because the equation (15) becomes

$$|a_{ij}' - \theta(a_{ij}' + b_{ij}')| = 0, \tag{15'}$$

where $a_{ij}' = \sum_{r=1}^{n_1} u_{ir}u_{jr}, \quad b_{ij}' = \sum_{r=1}^{n} v_{ir}v_{jr}, \quad (i, j = 1, 2, \ldots, p).$

To show this, we have, remembering the orthogonality of $\boldsymbol{\Gamma}_2$,

$$\|a_{ij}\| = \mathbf{YY}' = (\mathbf{C}')^{-1}\boldsymbol{\Gamma}_1'\mathbf{UU}'\boldsymbol{\Gamma}_1\mathbf{C}^{-1},$$
$$\|b_{ij}\| = \mathbf{ZZ}' = (\mathbf{C}')^{-1}\boldsymbol{\Gamma}_1'\mathbf{VV}'\boldsymbol{\Gamma}_1\mathbf{C}^{-1}.$$

Hence the equation (15) is transformed into

$$|\mathbf{UU}' - \theta(\mathbf{UU}' + \mathbf{VV}')| = 0,$$

which is another way of writing (15').

On the other hand, substituting (24) into (22) and remembering (23) we get the expression

$$-\tfrac{1}{2}\Psi - \tfrac{1}{2}\operatorname{tr}(\mathbf{UU}') - \tfrac{1}{2}\operatorname{tr}(\mathbf{VV}') + \operatorname{tr}(\mathbf{DU}')$$
$$= -\tfrac{1}{2}\Psi - \frac{1}{2}\sum_{i=1}^{p} a_{ii}' - \frac{1}{2}\sum_{i=1}^{p} b_{ii}' + \sum_{i=1}^{l}\sqrt{\lambda_i}\,u_{ii}.$$

Replacing again the letters u and v by y and z respectively, we obtain the result.

It may be noticed that

$$\Psi = \operatorname{tr}(\mathbf{CC}'\mathbf{GG}') = \operatorname{tr}(\mathbf{C}'\mathbf{GG}'\mathbf{C}) = \sum_{i=1}^{l}\lambda_i. \tag{25}$$

Remark. For the case of k samples (see Example on p. 221) Fisher (1938) has considered the problem of testing for the colinearity or coplanarity of the k populations. It can be proved that the hypothesis of colinearity (or coplanarity) states that all except one (or two) of the λ's vanish.

6. *Behaviour of V when H_0 is not necessarily true.* The Laplace transform of a probability density function $p(x)$ vanishing identically for $x < 0$, viz. the integral

$$\int_0^\infty e^{-sx}\,p(x)\,dx,$$

has the following property.

LEMMA 2. *Let $p_n(x)$, $(n=1,2,3,\ldots)$, and $p(x)$ be probability density functions vanishing identically for $x<0$, and such that*

$$\lim_{n\to\infty}\int_0^\infty e^{-\alpha x}p_n(x)\,dx=\int_0^\infty e^{-\alpha x}p(x)\,dx \tag{26}$$

for every $\alpha>0$. Then $\lim_{n\to\infty}\int_0^x p_n(u)\,du=\int_0^x p(u)\,du.$

Proof. The function $f_n(x)=p(x)-p_n(x)$ is summable in $(0,\infty)$ and, by (26),

$$\lim_{n\to\infty}\int_0^\infty e^{-\alpha x}f_n(x)\,dx=0 \quad \text{for every } \alpha>0. \tag{27}$$

The function $$g_n(z)=\int_0^\infty e^{-zx}f_n(x)\,dx$$

is an analytic function of z, regular at every z with a positive real part and uniformly bounded. By (27) $g_n(z)$ tends to a limit whenever z is real and positive. Hence by Vitali's theorem of convergence* $g_n(z)$ tends to a limit uniformly in the half-plane on the right of the imaginary axis. This limit is an analytic function regular at every z with a positive real part and vanishes whenever z is real and positive; therefore it vanishes identically. Hence

$$\lim_{n\to\infty}\int_0^\infty e^{-\alpha x+itx}p_n(x)\,dx=\int_0^\infty e^{-\alpha x+itx}p(x)\,dx \tag{28}$$

for all $\alpha>0$ and real t. Let

$$L_n(\alpha)=\int_0^\infty e^{-\alpha x}p_n(x)\,dx, \quad L(\alpha)=\int_0^\infty e^{-\alpha x}p(x)\,dx,$$

so that $$\lim_{n\to\infty}L_n(\alpha)=L(\alpha). \tag{29}$$

It follows from (28) and (29) that

$$\lim_{n\to\infty}\frac{1}{L_n(\alpha)}\int_0^\infty e^{-\alpha u+itu}p_n(u)\,du=\frac{1}{L(\alpha)}\int_0^\infty e^{-\alpha u+itu}p(u)\,du,$$

whence, by a well-known property of the characteristic function,

$$\lim_{n\to\infty}\frac{1}{L_n(\alpha)}\int_0^x e^{-\alpha u}p_n(u)\,du=\frac{1}{L(\alpha)}\int_0^x e^{-\alpha u}p(u)\,du,$$

whence $$\lim_{n\to\infty}\int_0^x e^{-\alpha u}p_n(u)\,du=\int_0^x e^{-\alpha u}p(u)\,du,$$

whence $$\lim_{\alpha\to+0}\lim_{n\to\infty}\int_0^x e^{-\alpha u}p_n(u)\,du=\int_0^x p(u)\,du.$$

The order of the above repeated limit can be interchanged because

$$\int_0^x e^{-\alpha u}p_n(u)\,du\to\int_0^x p_n(u)\,du$$

uniformly in n as $\alpha\to+0$. This completes the proof.

* Titchmarsh (1932), p. 168.

We shall now derive an expression for the Laplace integral $\mathscr{E}(\exp(-\alpha V))$. To simplify the notation we shall use a single letter, e.g. y to denote a set of variables, e.g. all the y_{ir}, when they figure as arguments of a function. We shall write dy for Πdy and a single integration sign for multiple integrals with respect to the variables from $-\infty$ to ∞. We denote the integral $\mathscr{E}(\exp(-\alpha V))$ by $L(\alpha)$.

The following equation can be verified by direct integration, on referring to (10) as the definition of V:

$$\exp(-\tfrac{1}{2}\alpha^2 V) = \int f(y,z,\alpha,t)\,dt,$$

where

$$f(y,z,\alpha,t) = (2\pi)^{-\frac{1}{2}pn_1}|b_{ij}|^{\frac{1}{2}n_1}\exp\left\{-\frac{1}{2}\sum_{i,j=1}^{p} b_{ij}s_{ij} + i\alpha\sum_{i=1}^{p}\sum_{r=1}^{n_1} y_{ir}t_{ir}\right\},$$

$$s_{ij} = \sum_{r=1}^{n_1} t_{ir}t_{jr} \quad (i,j=1,2,\ldots,p),$$

and where the letter i, when not figuring as an index, stands for $\sqrt{-1}$. Denoting by $p(y,z)$ the coefficient of $\Pi dy\,dz$ in (21), we have

$$L(\tfrac{1}{2}\alpha^2) = \int p(y,z)\,dy\,dz \int f(y,z,\alpha,t)\,dt = \int dt \int p(y,z)f(y,z,\alpha,t)\,dy\,dz, \quad (30)$$

as the change of the order of the integration is obviously legitimate for every real α.

By direct substitution it results

$$\int p(y,z)f(y,z,\alpha,t)\,dy\,dz = (2\pi)^{-\frac{1}{2}p(2n_1+n)}\exp(-\tfrac{1}{2}\Psi)f_1(\alpha,t)f_2(\alpha,t), \quad (31)$$

where

$$f_1(\alpha,t) = \int \exp\left\{-\frac{1}{2}\sum_{i=1}^{p} a_{ii} + \sum_{i=1}^{l}\sqrt{\lambda_i}\,y_{ii} + i\alpha\sum_{i=1}^{p}\sum_{r=1}^{n_1} t_{ir}y_{ir}\right\} dy, \quad (32)$$

$$f_2(\alpha,t) = \int |b_{ij}|^{\frac{1}{2}n_1}\exp\left\{-\frac{1}{2}\sum_{i=1}^{p} b_{ii} - \frac{1}{2}\sum_{i,j=1}^{p} s_{ij}b_{ij}\right\} dz. \quad (33)$$

The integral (32) is readily evaluated and runs

$$f_1(\alpha,t) = (2\pi)^{\frac{1}{2}pn_1}\exp(\tfrac{1}{2}\Psi)\exp\left\{-\tfrac{1}{2}\alpha^2\sum_{i=1}^{p}\sum_{r=1}^{n_1} t_{ir}^2 - i\alpha\sum_{i=1}^{l}\sqrt{\lambda_i}\,t_{ii}\right\}. \quad (34)$$

The integral (33) can be evaluated by using Wilks's formula for the moments of the generalized variance.* The result is

$$f_2(\alpha,t) = 2^{\frac{1}{2}pn_1}(2\pi)^{\frac{1}{2}pn}\,K\,|\delta_{ij}+s_{ij}|^{-\frac{1}{2}(n_1+n)}, \quad (35)$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, and where

$$K = \prod_{i=1}^{p}\frac{\Gamma\frac{1}{2}(n_1+n-i+1)}{\Gamma\frac{1}{2}(n-i+1)}.$$

* Wilks (1932).

From (34), (35), (31) and (30) we obtain

$$L(\tfrac{1}{2}\alpha^2) = \pi^{-\frac{1}{2}pn_1} K \int |\delta_{ij}+s_{ij}|^{-\frac{1}{2}(n_1+n)} \exp\left\{-\tfrac{1}{2}\alpha^2 \sum_{i=1}^{p} s_{ii} - i\alpha \sum_{i=1}^{l} \sqrt{\lambda_i}\, t_{ii}\right\} dt,$$

whence

$$L(\alpha) = \pi^{-\frac{1}{2}pn_1} K \int |\delta_{ij}+s_{ij}|^{-\frac{1}{2}(n_1+n)} \exp\left\{-\alpha \sum_{i=1}^{p} s_{ii} - i\sqrt{(2\alpha)} \sum_{i=1}^{l} \sqrt{\lambda_i}\, t_{ii}\right\} dt. \quad (36)$$

It may be observed that the right-hand side of (36) involves no sum of n terms; hence it is particularly useful when we wish to make n approach infinity.

We may calculate the moments of V by means of (36). Thus

$$\begin{aligned}
\mathscr{E}(V) &= \pi^{-\frac{1}{2}pn_1} K \int |\delta_{ij}+s_{ij}|^{-\frac{1}{2}(n_1+n)} \left\{\sum_{i=1}^{p} s_{ii} + \left(\sum_{i=1}^{l} \sqrt{\lambda_i}\, t_{ii}\right)^2\right\} dt \\
&= \frac{1}{n-p-1}(\Psi + pn_1), \qquad (37)
\end{aligned}$$

$$\begin{aligned}
\mathscr{E}(V^2) &= \pi^{-\frac{1}{2}pn_1} K \int |\delta_{ij}+s_{ij}|^{-\frac{1}{2}(n_1+n)} \left\{\left(\sum_{i=1}^{p} s_{ii}\right)^2 + 2\left(\sum_{i=1}^{p} s_{ii}\right)\left(\sum_{i=1}^{l} \sqrt{\lambda_i}\, t_{ii}\right)^2 \right. \\
&\qquad \left. + \frac{1}{3}\left(\sum_{i=1}^{l} \sqrt{\lambda_i}\, t_{ii}\right)^4\right\} dt \\
&= \frac{\Psi^2}{(n-p-1)(n-p-3)} - \frac{4\sum_{i\neq j} \lambda_i \lambda_j}{(n-p)(n-p-1)(n-p-3)} \\
&\quad + 2\Psi\left\{\frac{pn_1+2}{(n-p-1)(n-p-3)} - \frac{2(p-1)(n_1-1)}{(n-p)(n-p-1)(n-p-3)}\right\} \\
&\quad + \frac{pn_1(pn_1+2)}{(n-p-1)(n-p-3)} - \frac{2(p-1)(n_1-1)}{(n-p)(n-p-1)(n-p-3)}. \qquad (38)
\end{aligned}$$

Remark. Consider again the case of k samples (see Example on p. 221) and write V_k and Ψ_k correspondingly. It is seen from (37) that the statistic

$$(M-k-p-1)V_k - p(k-1)$$

is an unbiassed estimate of Ψ_k. We may regard V_k as a generalization of V_2, which is the "Studentized" D^2 statistic* but for a constant factor. V_2 is, of course, also identical with Hotelling's generalized "Student's" ratio except for a constant factor.

We shall next study the behaviour of V as $n \to \infty$. As n is now allowed to vary, we shall attach to various letters the index n. In particular we write Ψ_n for Ψ to emphasize the fact that the value of Ψ depends also on n, a fact which is not brought out by the formal definition of Ψ. This is because the η_{ir}, as we explained on p. 221, are themselves linear functions of the original population constants with coefficients depending upon n.

* Bose and Roy (1938).

It is in general true that, except when H_0 is true, in which case $\Psi_n \equiv 0$, Ψ_n is either $O(1)$ or $O(n)$ as $n \to \infty$. That both are possible is seen in the Example on p. 221 for $k = 2$; Ψ_n is $O(n)$ or $O(1)$ according as $m_1 = m_2 \to \infty$ or one of m_1, m_2 remains fixed while the other tends to infinity.

THEOREM 4. *If $\Psi_n = O(n)$, then the mean of V is $\frac{1}{n}\Psi_n + O\left(\frac{1}{n}\right)$ and the variance of V is $O\left(\frac{1}{n}\right)$. Hence the random variable $V - \frac{1}{n}\Psi_n \to 0$ in probability.*

This is an immediate consequence of (37) and (38).

If $\Psi_n = O(1)$, and if Ψ_n tends to a limit as $n \to \infty$, the limiting distribution of nV will be that of S (cf. (2)) with Ψ replaced by its limit. This is our next theorem. We assume now that

$$\lim_{n \to \infty} \Psi_n = \Psi_0. \tag{39}$$

The case $\Psi_n \equiv 0$ (i.e. H_0 is true) is covered by (39) on putting $\Psi_0 = 0$. But (39) may also be regarded as covering the case $\Psi_n = O(n)$ in the sense that the repeated sampling is not made from the same population, but from populations whose distribution constants vary with n in such a way that (39) holds true. Thus, for example, in the case of two samples with $m_1 = m_2$ (see p. 222) we have

$$\Psi_n = \tfrac{1}{2}m\Phi = \tfrac{1}{4}(n+2)\Phi,$$

where

$$\Phi = \sum_{i,j=1}^{p} \alpha_{ij}(\xi_{i1} - \xi_{i2})(\xi_{j1} - \xi_{j2}).$$

Here we have $\Psi_n = O(n)$ if the population constant $\Phi > 0$. Ψ_n will be $O(1)$ if we consider Φ as decreasing to the order n^{-1} as n increases indefinitely. This idea of regarding population constants as varying with the sample size can be found in the works of Fisher* and Neyman.†

THEOREM 5. *If Ψ_n tends to a limit Ψ_0 as $n \to \infty$, then*

$$\lim_{n \to \infty} Pr\{nV \leqslant x\} = \int_0^x p(x)\,dx, \tag{40}$$

where

$$p(x) = 2^{-\frac{1}{2}pn_1} x^{\frac{1}{2}pn_1 - 1} e^{-\frac{1}{2}(x + \Psi_0)} \sum_{h=0}^{\infty} \frac{\Psi_0^h x^h}{4^h h!\,\Gamma(h + \frac{1}{2}pn_1)}. \tag{41}$$

In particular, if $\Psi_0 = 0$ (i.e. H_0 is true), the limiting distribution of nV is that of χ^2 with pn_1 degrees of freedom.

Proof. Write $L_1^{(n)}(\alpha)$ for the Laplace transform of the probability density function of nV, so that

$$L_1^{(n)}(\alpha) = \mathscr{E}(\exp(-\alpha n V)) = L(n\alpha).$$

* Fisher (1928), p. 663.

† Neyman (1937), pp. 169–70, (1938), pp. 70–1.

If we replace α by $n\alpha$ in (36) and subsequently make the change of variables $t_{ir} = n^{-\frac{1}{2}}\tau_{ir}$, we get

$$L_1^{(n)}(\alpha) = \pi^{-\frac{1}{2}pn_1} n^{-\frac{1}{2}pn_1} K_n \int \left|\delta_{ij} + \frac{s'_{ij}}{n}\right|^{-\frac{1}{2}(n_1+n)} \exp\left\{-\alpha \sum_{i=1}^{p} s'_{ii} - i\sqrt{(2\alpha)} \sum_{i=1}^{l} \sqrt{\lambda_i}\,\tau_{ii}\right\} d\tau, \tag{42}$$

where K_n is now written for K and

$$s'_{ij} = \sum_{r=1}^{n_1} \tau_{ir}\tau_{jr}, \quad (i, j = 1, 2, \ldots, p).$$

We shall now find the limit of $L_1^{(n)}(\alpha)$ as $n \to \infty$. We have, using Stirling's formula,

$$\lim_{n\to\infty} n^{-\frac{1}{2}pn_1} K_n = 2^{-\frac{1}{2}pn_1}. \tag{43}$$

For every system of fixed values of the τ_{ir} we have

$$\left|\delta_{ij} + \frac{s'_{ij}}{n}\right| = 1 + \frac{1}{n}\sum_{i=1}^{p} s'_{ii} + O\left(\frac{1}{n^2}\right),$$

whence

$$\lim_{n\to\infty} \left|\delta_{ij} + \frac{s'_{ij}}{n}\right|^{\frac{1}{2}(n_1+n)} = \exp\left(-\frac{1}{2}\sum_{i=1}^{p} s'_{ii}\right). \tag{44}$$

Calling I_n the integral in (42), we have

$$\begin{aligned} I_n = \int \left\{ \left|\delta_{ij} + \frac{s'_{ij}}{n}\right|^{-\frac{1}{2}(n_1+n)} - \exp\left(-\tfrac{1}{2}\textstyle\sum s'_{ii}\right) \right\} \exp\left\{-\alpha \sum_{i=1}^{p} s'_{ii} - i\sqrt{(2\alpha)} \sum_{i=1}^{l} \sqrt{\lambda_i}\,\tau_{ii}\right\} d\tau \\ + \int \exp\left\{-(\tfrac{1}{2}+\alpha) \sum_{i=1}^{p} s_{ii} - i\sqrt{(2\alpha)} \sum_{i=1}^{l} \sqrt{\lambda_i}\,\tau_{ii}\right\} d\tau \\ = A_n + B_n. \end{aligned} \tag{45}$$

The absolute value of the integrand of A_n is less than

$$2\exp\left(-\alpha \sum_{i=1}^{p} s'_{ii}\right).$$

Hence by (44) and dominated convergence, $A_n \to 0$ as $n \to \infty$. The value of B_n is easily found to be

$$B_n = (2\pi)^{\frac{1}{2}pn_1} (1+2\alpha)^{-\frac{1}{2}pn_1} \exp\left(-\frac{\alpha\Psi_n}{1+2\alpha}\right).$$

From (42), (43) and (45) we obtain the result

$$\lim_{n\to\infty} L_1^{(n)}(\alpha) = (1+2\alpha)^{-\frac{1}{2}pn_1} \exp\left(-\frac{\alpha\Psi_0}{1+2\alpha}\right). \tag{46}$$

If $p(x)$ is defined as in (41), then the integral

$$\int_0^\infty e^{-\alpha x} p(x)\, dx$$

is identically equal to the right-hand side of (46). Theorem 5 is thus proved on remembering Lemma 2.

7. *Behaviour of W when H_0 is not necessarily true and n is large.* We shall establish the following theorem.

THEOREM 6. *If $\Psi_n = O(1)$, the statistics nV and $-n\log W$ tend to be certainly identical in the sense that*

$$\lim_{n\to\infty}(nV + n\log W) = 0 \text{ in probability.} \tag{47}$$

Corollary. If Ψ_n tends to a limit Ψ_0 as $n\to\infty$, $-n\log W$ has the same limiting distribution (40) *as nV as $n\to\infty$.*

Remark. Besides the Corollary there is another significance to be attached to Theorem 6. Since the test functions V and W are not functionally related (except when $p = 1$ or $n_1 = 1$), there is the question of choosing one of them to be consistently used in carrying out the actual tests. This can only be decided by a comparison of their power* to detect the falsehood of H_0 when the η_{ir} do not all vanish. While this may be a difficult problem for small samples, Theorem 6 appears to have answered the question for large samples. In fact, if n is large, it is almost certain that the values of nV and $-n\log W$ calculated from the sample will differ very little. That $-n\log W$ and nV tend to have the same power function is a consequence of the Corollary.

Proof of Theorem 6. For every θ such that $0\leqslant\theta\leqslant 1$ we have

$$0\leqslant\frac{\theta}{1-\theta}+\log(1-\theta)\leqslant\frac{1}{2}\left(\frac{\theta}{1-\theta}\right)^2.$$

Hence, remembering (12) and (13),

$$0\leqslant V+\log W\leqslant\frac{1}{2}\sum_{i=1}^{l_1}\left(\frac{\theta_i}{1-\theta_i}\right)^2\leqslant\frac{1}{2}\left(\sum_{i=1}^{l_1}\frac{\theta_i}{1-\theta_i}\right)^2=\tfrac{1}{2}V^2,$$

whence, for every $\eta>0$,

$$Pr\{|\,nV+n\log W\,|>\eta\}=Pr\{nV+n\log W>\eta\}\leqslant Pr\left\{\frac{n}{2}V^2>\eta\right\}$$
$$\leqslant\frac{1}{\eta}\mathscr{E}\left(\frac{n}{2}V^2\right)=O\left(\frac{1}{n}\right)$$

because of (38). This establishes (47).

Proof of the Corollary. Let

$$F_n(x)=Pr\{nV\leqslant x\},\quad G_n(x)=Pr\{-n\log W\leqslant x\}.$$

Then for every x and $\eta>0$ we have†

$$0\leqslant G_n(x)-F_n(x)\leqslant F_n(x+\eta)-F_n(x-\eta)+Pr\{nV+n\log W\geqslant\eta\}.$$

* Neyman & Pearson (1936, 1938).

† Fréchet (1937), p. 164.

Let $F(x)$ be the limit of $F_n(x)$ as $n \to \infty$, which is known to exist by virtue of Theorem 6. We have then

$$0 \leqslant G_n(x) - F_n(x) \leqslant |F_n(x+\eta) - F(x+\eta)| + |F_n(x-\eta) - F(x-\eta)| + F(x+\eta) - F(x-\eta) + Pr\{nV + n \log W \geqslant \eta\}. \quad (48)$$

Given any $\epsilon > 0$, choose and fix an $\eta > 0$ so small that $F(x+\eta) - F(x-\eta) < \epsilon$. By (47) the rest of the terms in the right-hand side of (48) is smaller than ϵ for all sufficiently large n. Hence $G_n(x) - F_n(x) \to 0$, i.e. $G_n(x) \to F(x)$, which completes the proof.

Summary

The Wilks-Lawley hypothesis concerning population means of multivariate normal populations is put in the canonical form H_0 (§ 1), and, assuming the population variances and covariances known, the test function S is derived together with its exact distribution (§ 2). In the case where the population variances and covariances are unknown, two possible test functions, denoted by V and W are considered (§§ 3 and 4), and their distributions in certain special cases are given. In § 5 it is shown that the sample space can be so transformed that all the variables are independently distributed and that only a minimum number of unknown parameters remain. These parameters are the roots of a certain determinantal equation; the hypothesis H_0, and the hypotheses of colinearity and coplanarity of populations, all specify the values zero for all or some of these parameters. Returning to the test functions V and W, it is shown that as the sample size n increases indefinitely the two functions nV and $-n \log W$ tend to be certainly identical ((47), § 7), and both of them tend to have the same distribution function as S ((40), § 6).

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THE DERIVATION OF THE FIFTH AND SIXTH MOMENTS OF THE DISTRIBUTION OF b_2 IN SAMPLES FROM A NORMAL POPULATION

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1. Introduction

The method of combinatorial analysis was first introduced in 1928 in a paper by R. A. Fisher (1929). In this paper Fisher defined new symmetric functions $k_1, k_2, k_3, \ldots$ of the observations for samples of a given size, and gave simple rules for determining the cumulants or semi-invariants of their joint sampling distribution. This method is especially valuable for the case where the sampled population is normal, and in this case its use reduces considerably the labour of deriving the higher sampling moments of the distribution of product moment statistics. Two further papers appeared later, one a joint paper of R. A. Fisher and J. Wishart (1931) in which further rules were given, the other by Wishart (1930) in which he described applications of the theory and gave a list of higher order formulae for the normal case. By means of these formulae E. S. Pearson (1930) was able to derive the first four moments of the sampling distribution of the statistics $\sqrt{b_1}$ and b_2 for the case when the sampled population was assumed to be normal.

The object of the present paper is to derive formulae for $\kappa(4^5)$ and $\kappa(4^6)$ (quantities which are defined below) and then to use these to determine the fifth and sixth moment coefficients of the distribution of b_2. This work is preliminary to further investigations regarding the sampling distribution of this expression.

If $x_1, x_2, \ldots, x_n$ be a sample of size n from a given population, then adopting Fisher's definition of the symmetric functions k_1, k_2, k_3, k_4, we have

$$k_1 = m_1,$$

$$k_2 = \frac{n}{n-1} m_2,$$

$$k_3 = \frac{n^2}{(n-1)(n-2)} m_3,$$

$$k_4 = \frac{n^2}{(n-1)(n-2)(n-3)} \{(n+1) m_4 - 3(n-1) m_2^2\},$$

where

$$m_r = \frac{1}{n} \overset{n}{S}_1 (x - \bar{x})^r, \quad \bar{x} = \frac{1}{n} \overset{n}{S}_1 x.$$

The coefficients k_r are chosen so that $E(k_r) = \kappa_r$, where κ_r is the rth cumulant of the given population, and the pth cumulant of the distribution of k_r is as usual denoted by $\kappa(r^p)$.

$\kappa(r^p)$ is expressible as the sum of a number of terms of order pr consisting of powers and products of the coefficients $\kappa_2, \kappa_3, \ldots, \kappa_{pr}$, and to each such term there corresponds a number of two-way partitions whose coefficients have to be evaluated. As in this paper we are, however, assuming the sampled population to be normal, the only non-vanishing cumulant is κ_2, which is equal to σ^2, the variance of the distribution. The only term, therefore, which has to be evaluated is the one containing a power of κ_2.

2. THE DERIVATION OF $\kappa(4^5)$

For a full explanation of what follows the reader is referred to the three papers by Fisher and Wishart already cited.

To determine the formula for $\kappa(4^5)$ we must find the coefficients of all the two-way patterns for the term in κ_2^{10}. There are altogether five such patterns, which are given in Wishart's paper, and they are reproduced below. In each pattern it will be noted that there are five corners with four arms attached, each corner representing a k_4, and ten connections between the pairs of arms, each connection representing a κ_2.

Fig. 1.

For each of these five patterns, we must determine (1) the numerical coefficient, (2) the n-coefficient.

It is not necessary to give in detail all the working required for doing this, but we shall give two examples for each process and also a summary of the results.

(1) The numerical coefficient is obtained by enumerating all the ways in which the pattern can be connected up, regarding as a separate entity each corner and also each arm attached to that corner.

(a) Consider first the pattern A.

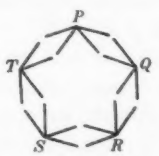

Fig. 2.

There are $\frac{1}{2} \times 4! = 12$ ways of determining the cyclic order of the five corners.

There are six ways of choosing a pair of arms belonging to corner P to connect up to Q, and similarly for the other four corners.

Finally, there are two ways of joining up the double arms between each pair of adjacent corners. The numerical coefficient of A is therefore $12 \times 6^5 \times 2^5 = 12 \times 12^5$.

(*b*) Consider the pattern D.

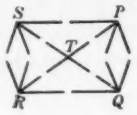

Fig. 3.

There are five ways of choosing which corner is to form the centre T.

There are three ways of choosing the pairs of corners P, Q and R, S which are linked up by double arms, and two ways of connecting up these two pairs.

The arms of corner T may be arranged in $4! = 24$ ways, while those of each of the other corners may be arranged in 12 ways.

Finally, there are two ways of connecting up each of the two double arms.

Hence the numerical coefficient* of D is

$$5 \times (3 \times 2) \times (24 \times 12^4) \times 2^2 = 240 \times 12^5.$$

We may remark that the five normal patterns for $\kappa(4^5)$ can all be obtained from those of $\kappa(4^4)$ by the insertion of a fifth corner. This fact allows us to derive the numerical coefficients by a different method, which may be used as a check. This method must, however, be employed with care as otherwise it is liable to suggest a wrong result. We shall illustrate it by considering again the pattern A.

The pattern A may be formed from the smaller pattern which has a numerical coefficient of 62208, by breaking one of the double arms and inserting a new four-way corner.

Fig. 4.

The break may be made in four places and the arms of the new corner may then be joined up in 24 ways. The numerical coefficient of A would thus at first sight appear to be $62208 \times 4 \times 24 = 24 \times 12^5$.

It must, however, be remembered that when for instance the break in the smaller pattern is made between P and Q, the broken pattern corresponds to two arrangements of the unbroken pattern, since there are two ways of joining up the double arms between P and Q. Hence the true value of the numerical coefficient of A is in fact half the above number, i.e. 12×12^5.

* There is apparently a mistake in Wishart's paper, the numerical coefficient of D being given there as 120×12^5. This will of course mean that the approximate value of $\kappa(4^5)$ found in that paper is incorrect.

(2) The n-coefficient.

In a previous paper (Fisher & Wishart, 1931) rules were given for deriving the n-coefficients of a given pattern from those of non-vanishing patterns of a lower order which are already known. Thus we can derive the n-coefficient of A, which is one of the patterns of $\kappa(4^5)$, from those of two of the patterns of $\kappa(4^4)$. This is illustrated in Fig. 5, the Greek letters below each pattern are used to denote the n-coefficient of that pattern.

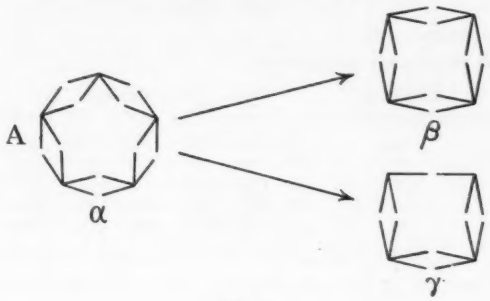

Fig. 5.

Using the rule for adding a new k_4, we have

$$\alpha = 2\,\frac{(-1)}{(n-2)(n-3)}\,\beta + \frac{n(n+1)}{(n-1)(n-2)(n-3)}\,\gamma,$$

and we also know that

$$\beta = \frac{n(n+1)\,n^4 - 8n^3 + 21n^2 - 14n + 4}{(n-1)^3(n-2)^3(n-3)^3}$$

and

$$\gamma = \frac{n^3}{(n-1)^3(n-2)^3}.$$

Hence we obtain the result

$$\alpha = \frac{n(n+1)}{(n-1)^4(n-2)^4(n-3)^4}\{n^6 - 11n^5 + 45n^4 - 85n^3 + 70n^2 - 36n + 8\}.$$

This is the n-coefficient of A.

Similarly for pattern B.

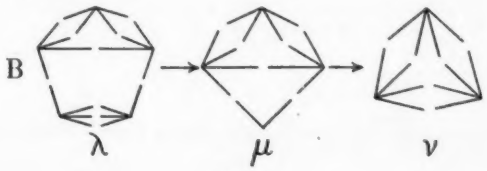

Fig. 6.

Here we have $$\lambda = \frac{n+1}{(n-1)(n-2)(n-3)}\mu,$$

$$\mu = \frac{n}{n-1}\nu$$

and $$\nu = \frac{n(n+1)(n^2-5n+2)}{(n-1)^4(n-2)^3(n-1)^3}.$$

Hence $$\lambda = \frac{n^2(n+1)^2(n^2-5n+2)}{(n-1)^4(n-2)^3(n-3)^3}.$$

This is the n-coefficient of B.

The complete results are as follows:

Pattern	Numerical coefficient/12^5	n-coefficient $\div n(n+1)/\{(n-1)^4(n-2)^4(n-3)^4\}$
A	12	$n^6-11n^5+45n^4-85n^3+70n^2-36n+8$
B	80	$n^6-9n^5+23n^4-7n^3-28n^2+12n$
C	120	$n^6-12n^5+52n^4-94n^3+59n^2-30n+8$
D	240	$n^6-13n^5+59n^4-103n^3+48n^2-24n+8$
E	32	$n^6-15n^5+80n^4-160n^3+75n^2-41n+12$

The formula for $\kappa(4^5)$ is calculated by finding the sum of products of the numerical coefficients with the n-coefficients. We thus obtain the result

$$\kappa(4^5) = \frac{4\times 12^5 n(n+1)}{(n-1)^4(n-2)^4(n-3)^4}\{121n^6-1473n^5+6335n^4 - 10{,}675n^3+4900n^2-2536n+840\}\,\kappa_2^{10}.$$

3. The derivation of $\kappa(4^6)$

There are altogether 17 normal patterns for $\kappa(4^6)$, which are shown in Fig. 7.

All the patterns except the last two (R and S) may be obtained from those of $\kappa(4^5)$ by the insertion of a sixth corner, while the patterns R and S may be obtained from those of $\kappa(4^4)$ by the insertion of a pair of corners. The numerical coefficients may as before be calculated in two ways and we shall content ourselves with a summary of the results. We shall, however, give two examples of the derivation of the n-coefficients. That for the pattern A is shown in Fig. 8. Here we have

$$\alpha = 2\times\frac{(-1)}{(n-2)(n-3)}\beta + \frac{n(n+1)}{(n-1)(n-2)(n-3)}\gamma,$$

$$\beta = \frac{n(n+1)}{(n-1)^4(n-2)^4(n-3)^4}\{n^6-11n^5+45n^4-85n^3+70n^2-36n+8\},$$

$$\gamma = \frac{n}{(n-1)(n-2)}\delta = \frac{n}{(n-1)(n-2)}\times\frac{n^3}{(n-1)^3(n-2)^3} = \frac{n^4}{(n-1)^4(n-2)^4}.$$

Hence $$\alpha = \frac{n(n+1)}{(n-1)^5(n-2)^5(n-3)^5}\{n^8-14n^7+78n^6-220n^5+341n^4 - 310n^3+212n^2-88n+16\}.$$

This is the n-coefficient of A.

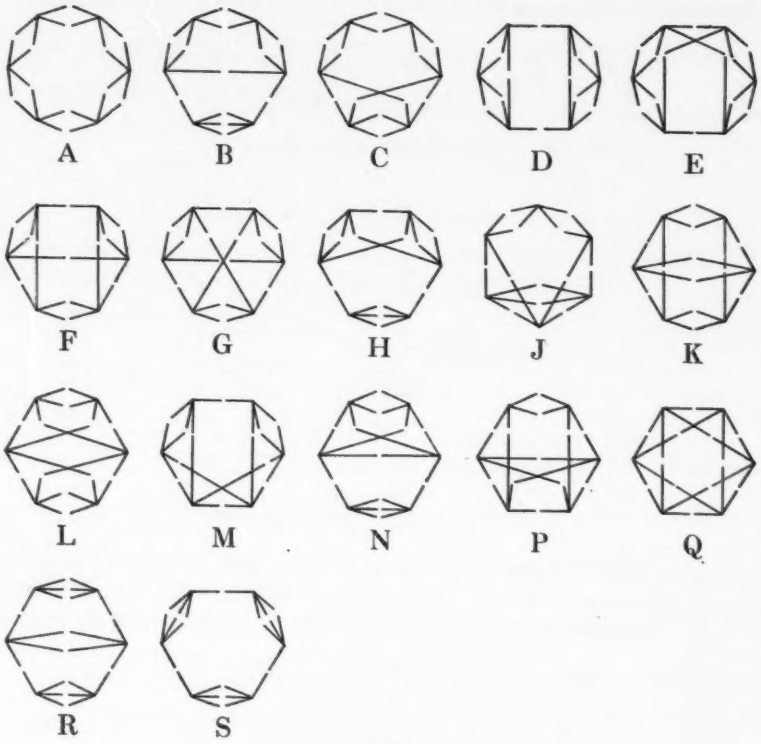

Fig. 7.

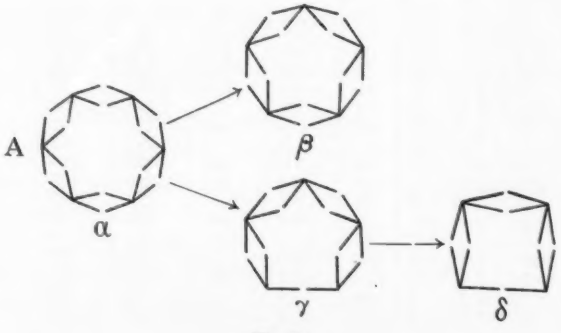

Fig. 8.

Similarly for the pattern S.

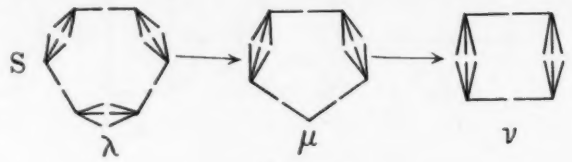

Fig. 9.

$$\lambda = \frac{(n+1)}{(n-1)(n-2)(n-3)}\mu,$$

$$\mu = \frac{n}{n-1}\nu,$$

and

$$\nu = \frac{n^2(n+1)^2}{(n-1)^3(n-2)^2(n-3)^2}.$$

Hence the n-coefficient of S is

$$\lambda = \frac{n^3(n+1)^3}{(n-1)^5(n-2)^3(n-3)^3}.$$

The complete results are given in the following table. Here c denotes the numerical coefficient divided by k, where $k = 2^{14} \times 3^4 \times 5 = 6{,}635{,}520$ and a_r is the coefficient of n^r in the expression for n-coefficient divided by N, where

$$N = \frac{n(n+1)}{(n-1)^5(n-2)^5(n-3)^5}.$$

Table of coefficients

Pattern	c	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0
A	27	1	−14	78	−220	341	−310	212	− 88	16
B	216	1	−12	54	−100	33	96	− 80	24	0
C	324	1	−15	88	−254	373	−279	178	− 76	16
D	324	1	−14	70	−140	53	138	−116	24	0
E	162	1	−15	90	−266	389	−267	160	−176	16
F	648	1	−16	99	−292	403	−236	153	− 64	16
G	216	1	−16	104	−324	449	−208	110	− 52	16
H	432	1	−13	60	−106	17	111	− 62	24	0
J	1296	1	−16	100	−300	421	−236	126	− 64	16
K	216	1	−16	103	−320	451	−220	101	− 52	16
L	324	1	−16	99	−292	403	−236	153	− 64	16
M	2592	1	−17	111	−338	451	−193	101	− 52	16
N	432	1	−13	60	−106	17	111	− 62	24	0
P	1296	1	−18	129	−442	675	−342	195	− 94	24
Q	432	1	−19	144	−521	825	−385	210	−107	28
R	72	1	− 8	18	4	− 47	12	36	0	0
S	16	1	− 8	18	4	− 47	12	36	0	0

The formula for $\kappa(4^6)$ derived from these results is

$$\kappa(4^6) = 6{,}635{,}520\frac{n(n+1)}{(n-1)^5(n-2)^5(n-3)^5}\{9025n^8 - 145{,}532n^7 + 920{,}610n^6 - 2{,}775{,}248n^5 + 3{,}759{,}853n^4 - 1{,}717{,}116n^3 + 946{,}872n^2 - 476{,}064n + 136{,}080\}\kappa_2^{12}.$$

4. THE MOMENTS OF THE DISTRIBUTION OF b_2

The formulae of $\kappa(4^2)$, $\kappa(4^3)$ and $\kappa(4^4)$ have been found previously and we will give the results:

$$\kappa(4^2) = \frac{24n(n+1)}{(n-1)(n-2)(n-3)}\kappa_2^4,$$

$$\kappa(4^3) = \frac{1728n(n+1)(n^2-5n+2)}{(n-1)^2(n-2)^2(n-3)^2}\kappa_2^6,$$

$$\kappa(4^4) = \frac{6912n(n+1)(53n^4-428n^3+1025n^2-474n+180)}{(n-1)^3(n-2)^3(n-3)^3}\kappa_2^8.$$

We may now calculate the moments of the distribution of k_4 as follows:

$$\mu(4^2) = \kappa(4^2),$$

$$\mu(4^3) = \kappa(4^3),$$

$$\mu(4^4) = \kappa(4^4) + 3\kappa^2(4^2) = \frac{1728n(n+1)}{(n-1)^3(n-2)^3(n-3)^3}\{n^5 + 207n^4 - 1707n^3 + 4105n^2 - 1902n + 720\}\kappa_2^8,$$

$$\mu(4^5) = \kappa(4^5) + 10\kappa(4^2)\kappa(4^3) = \frac{12^4 \cdot 4n(n+1)}{(n-1)^4(n-2)^4(n-3)^4}\{5n^7 + 1402n^6 - 17{,}516n^5 + 75{,}870n^4 - 128{,}205n^3 + 59{,}000n^2 - 30{,}492n + 10{,}080\}\kappa_2^{10},$$

$$\mu(4^6) = \kappa(4^6) + 15\kappa(4^4)\kappa(4^2) + 10\kappa^2(4^3) + 15\kappa^3(4^2) = \frac{12^4 \cdot 10n(n+1)}{(n-1)^5(n-2)^5(n-3)^5}\{n^{10} + 770n^9 + 278{,}359n^8 - 4{,}603{,}808n^7 + 29{,}339{,}555n^6 - 88{,}717{,}430n^5 + 120{,}300{,}577n^4 - 55{,}075{,}788n^3 + 30{,}365{,}028n^2 - 15{,}250{,}464n + 4{,}354{,}560\}\kappa_2^{12}.$$

The quantity b_2 is defined to be m_4/m_2^2 and, using the definitions of k_2 and k_4 given in § 1, we see that

$$b_2 = \frac{(n-1)(n-2)(n-3)}{n^2(n+1)}\frac{k_4}{m_2^2} + \frac{3(n-1)}{n+1} = \frac{(n-2)(n-3)}{(n-1)(n+1)}\frac{k_4}{k_2^2} + \frac{3(n-1)}{n+1}.$$

Thus the average value of b_2 is $3(n-1)/(n+1)$ and, for $r>1$, the rth moment of the distribution of b_2 is given by

$$\mu_r(b_2) = \left\{\frac{(n-2)(n-3)}{(n-1)(n+1)}\right\}^r \mu_r(k_4 k_2^{-2})$$

$$= \left\{\frac{(n-2)(n-3)}{(n-1)(n+1)}\right\}^r \mu(4^r 2^{-r}).$$

Now Fisher (1930) has shown that

$$\mu(\dots 5^c 4^b 3^a) = \mu(\dots 5^c 4^b 3^a 2^{-r}) \frac{(n-1)\dots(n+2r-3)}{(n-1)^r} \kappa_2^r,$$

and in particular

$$\mu(4^r) = \mu(4^r 2^{-2r}) \frac{(n-1)(n+1)\dots(n+4r-3)}{(n-1)^{2r}} \kappa_2^{2r}.$$

Thus
$$\mu_r(b_2) = \frac{(n-1)^r (n-2)^r (n-3)^r}{(n+1)^r (n-1)(n+1)\dots(n+4r-3)} \frac{\mu(4^r)}{\kappa_2^{2r}}.$$

Hence, finally, we obtain the following results:

$$\mu_2(b_2) = \frac{24n(n-2)(n-3)}{(n+1)^2(n+3)(n+5)},$$

$$\mu_3(b_2) = \frac{1728n(n-2)(n-3)(n^2-5n+2)}{(n+1)^3(n+3)(n+5)(n+7)(n+9)},$$

$$\mu_4(b_2) = \frac{1728n(n-2)(n-3)(n^5+207n^4-1707n^3+4105n^2-1902n+720)}{(n+1)^4(n+3)(n+5)(n+7)(n+9)(n+11)(n+13)},$$

$$\mu_5(b_2) = \frac{12^4 . 4n(n-2)(n-3)}{(n+1)^5(n+3)(n+5)\dots(n+17)}\{5n^7+1402n^6-17{,}516n^5+75{,}870n^4$$
$$-128{,}205n^3+59{,}000n^2-30{,}492n+10{,}080\},$$

$$\mu_6(b_2) = \frac{12^4 . 10n(n-2)(n-3)}{(n+1)^6(n+3)(n+5)\dots(n+21)}\{n^{10}+770n^9+278{,}359n^8-4{,}603{,}808n^7$$
$$+29{,}339{,}555n^6-88{,}717{,}430n^5+120{,}380{,}577n^4-55{,}075{,}788n^3$$
$$+30{,}365{,}028n^2-15{,}250{,}464n+4{,}354{,}560\}.$$

5. Check with McKay's results in the case $n = 4$

Since the formulae which we have obtained are somewhat complicated, it is desirable to provide some sort of check on them. This is supplied by using the results obtained by McKay (1933), who found in an exact form the distribution of b_2 for $n = 4$, i.e. for samples of size 4.

Putting $x = b_2$, the distribution function $f(x)$, of x, is given by

$$f(x) = \frac{3}{2^{\frac{1}{2}}(7-3x)^{\frac{1}{2}}} F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{1-g}{2}\right) \quad \text{when } 1 \leqslant x \leqslant 2,$$

$$= \frac{3}{2(9x-17)} F\left(\tfrac{1}{4}, \tfrac{1}{4}; 1; \frac{1}{g^2}\right) \quad \text{when } 2 \leqslant x \leqslant \tfrac{7}{3},$$

where $$g = \frac{9x-17}{(7-3x)^{\frac{1}{2}}}.$$

The moments of $w = \frac{1}{4}b_2 - \frac{1}{4}$ about the origin are given by

$$\nu_k(w) = \frac{1}{2\Gamma\left(\frac{4k+3}{2}\right)} \left\{ \Gamma(k+\tfrac{1}{2})\,\Gamma(k+1) + \frac{1}{2}\frac{k\Gamma(k-\frac{1}{2})\,\Gamma(k+2)}{4(1!)^2} + \frac{1}{2}\cdot\frac{3}{2}\frac{k(k-1)\,\Gamma(k-\frac{3}{2})\,\Gamma(k+3)}{4^2(2!)^2} + \ldots \right\}.$$

Hence

$$\nu_1(w) = \frac{1}{5},$$

$$\nu_2(w) = \frac{1}{3.7} = \frac{1}{21},$$

$$\nu_3(w) = \frac{61}{5.7.11.13},$$

$$\nu_4(w) = \frac{277}{5.7.11.13.17},$$

$$\nu_5(w) = \frac{79}{3.7.13.17.19},$$

$$\nu_6(w) = \frac{46{,}889}{5.7.11.13.17.19.23}.$$

The moments about the mean of w are given by

$$\mu_2(w) = \frac{4}{3.5^2.7} = \frac{2^2}{3.5^2.7},$$

$$\mu_3(w) = \frac{-48}{5^2.7.11.13} = \frac{-3.2^4}{5^2.7.11.13},$$

$$\mu_4(w) = \frac{1424}{5^4.7.11.13.17} = \frac{2^4.89}{5^4.7.11.13.17},$$

$$\mu_5(w) = \frac{-42{,}624}{5^5.3.17.19} = \frac{-2^7.3.37}{5^5.17.19},$$

$$\mu_6(w) = \frac{76{,}096}{5^5.7.11.13.17.19.23} = \frac{2^6.1189}{5^5.7.11.13.17.19.23}.$$

Thus for the moments of b_2, we have

$$\mu_2(b_2) = 4^2\mu_2(w) = \frac{2^6}{3.5^2.7},$$

$$\mu_3(b_2) = 4^3\mu_3(w) = \frac{-2^{10}.3}{5^3.7.11.13},$$

$$\mu_4(b_2) = 4^4\mu_4(w) = \frac{2^{12}.89}{5^4.7.11.13.17},$$

$$\mu_5(b_2) = 4^5\mu_5(w) = \frac{-2^{17}.3.37}{5^5.7.11.13.17.19},$$

$$\mu_6(b_2) = 4^6\mu_6(w) = \frac{2^{18}.1189}{5^5.7.11.13.17.19.23}.$$

It may be easily verified that the same results are obtained by putting $n = 4$ in the formulae given at the end of the preceding section. There is thus reason to believe that these formulae are in fact correct.

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TESTING THE HOMOGENEITY OF A SET OF VARIANCES

By H. O. HARTLEY

1. Introduction

When analysing data the experimenter is frequently faced with the necessity of testing the homogeneity in a set of estimated variances. When it is desired to combine a number of variances to obtain an estimate of the common variance it is necessary to apply such a test. Again, if a selected "treatment mean square" is to be compared with an "error mean square", a test for homogeneity has recently been proposed (Wishart, 1938) as a safeguard against the selection of the largest mean square from a set of random ones.

For general use in such cases Neyman & Pearson (1931) have suggested a test: the L_1 test. The statistic L_1 used in this test has been modified by Bartlett (1937) and generalized by Welch (1935, 1936). From recent work (Nair, 1938; Bishop & Nair, 1939; Pitman, 1939) it would appear that Bartlett's statistic μ is the best to use, because it is unbiased in the sense defined by Neyman & Pearson (1936, 1938). Or more precisely, the L_1 test in its original form is biased with regard to the admissible set of alternatives.

Some difficulty has been experienced in obtaining the random sampling distribution of this statistic which is required for a test. Various approximations have been worked out. There are

(*a*) Bartlett's (1937) approximation using the χ^2 distribution.

(*b*) P. P. N. Nayer's (1936) approximation obtained by fitting Pearson-type curves to the distribution in the special case where all mean squares are based on the same number of degrees of freedom.

(*c*) U. S. Nair's (1938) expansion of the exact distribution in the special case mentioned in (*b*).

(*d*) Recently another paper on the subject has appeared by E. J. G. Pitman (1939). In this paper the author transforms the distribution of L_1 into a multiple integral which can be evaluated in special cases (small values of k) by reduction to elliptic integrals.

The accuracy of the approximations (*a*) and (*b*) has recently been tested (Bishop & Nair, 1939) in the special case in which the expansion (*c*) is available. Bartlett's findings were confirmed; it was shown that his approximation is valid only for moderate or large numbers of degrees of freedom ($\geqslant 3$). We shall also show in this paper that even with this restriction for the degrees of freedom the approximation is not very accurate if k, the number of mean squares in the set, is large.

While U. S. Nair's expansion, although it is very complicated, provides a means of working out the exact probability integral in the special case where all mean squares are based on the same number of degrees of freedom, there is still uncertainty in the general case. P. P. N. Nayer has suggested that the test for homogeneity between k mean squares with f_t degrees of freedom ($t = 1, 2, \ldots, k$) is (under certain conditions) identical with testing the homogeneity between k mean squares *all of which have f degrees of freedom*, where f is the arithmetic mean of the f_t. We shall show that, although there is some truth in this statement, the harmonic mean should be used for f rather than the arithmetic mean.

Since Bartlett's approximation does not provide a test of sufficient accuracy in all cases, the main difficulty of dealing with the general case has been the large number of quantities on which the exact distribution depends: if the k mean squares in the set have f_t degrees of freedom respectively ($t = 1, 2, \ldots, k$) the distribution would depend on $k+1$ quantities. We shall now show in this paper that (provided $f_t \geqslant 2$) there is an approximation of sufficient accuracy which depends on *three* quantities only. These three are:

(i) k, the number of mean squares in the set;

(ii) $c_1 = \sum_{t=1}^{k} \frac{1}{f_t} - \frac{1}{F}$, where $F = \sum_{t=1}^{k} f_t$;

(iii) $c_3 = \sum_{t=1}^{k} \frac{1}{f_t^3} - \frac{1}{F^3}$.

This makes the distribution amenable to tabulation, so that the test can be reduced to an inspection of a table of 5 % and 1 % points which can easily be carried out by the experimenter.

In the case where mean squares having one degree of freedom occur in the set, the distribution is of a more complicated character, but our approximation is still fair.

2. The formal solution

Consider k normal populations with variances σ_t^2 ($t = 1, 2, \ldots, k$). Let s_t^2 be an unbiased estimate of σ_t^2 based on f_t degrees of freedom, and let us denote by F the total number of degrees of freedom,

$$F = \sum_{t=1}^{k} f_t. \qquad \ldots\ldots(1)$$

Bartlett's statistic μ is then given by

$$-2 \log \mu = F \log \{\textstyle\sum_t (f_t s_t^2)/F\} - \sum_t f_t \log s_t^2. \qquad \ldots\ldots(2)$$

The equivalence to a special case of the generalized L_1 statistic (Welch, 1935, 1936) is expressed by the relation

$$F \log L_1' = 2 \log \mu, \qquad \ldots\ldots(3)$$

where
$$L_1' = \prod_{t=1}^{k} \left(\frac{F}{f_t}\right)^{f_t/F} \prod_{t=1}^{k} \left\{\frac{f_t s_t^2}{\sum_t f_t s_t^2}\right\}^{f_t/F}. \qquad \text{.....(4)}$$

For our test we require the random sampling distribution $\phi(L_1')$ of the statistic L_1' under the null hypothesis

$$\sigma_t^2 = \sigma^2, \quad t = 1, 2, \ldots, k.$$

Under these conditions it has recently been shown (Welch, 1936) that the $(q-1)$th sampling moment of L_1' is given by

$$\begin{aligned} M_{q-1} &= \int_0^1 \phi(L_1')\, L_1'^{q-1}\, dL_1' \\ &= \prod_{t=1}^{k} \left(\frac{F}{f_t}\right)^{((q-1)f_t)/F} \frac{\Gamma(\frac{1}{2}F)}{\Gamma(\frac{1}{2}F+q-1)} \prod_{t=1}^{k} \left\{\frac{\Gamma(\frac{1}{2}f_t+\{(q-1)f_t\}/F)}{\Gamma(\frac{1}{2}f_t)}\right\}. \qquad \text{.....(5)} \end{aligned}$$

From general principles it may now be inferred that equation (5) is valid for all complex q with

Real Part of $q > 1$.

Further, by Mellin's inversion formula, we obtain from (5)

$$\begin{aligned} \phi(L_1') &= \Gamma(\tfrac{1}{2}F) \prod_{t=1}^{k} \Gamma(\tfrac{1}{2}f_t)^{-1} \\ &\times \frac{1}{2\pi i} \int_{Q-i\infty}^{Q+i\infty} \prod_{t=1}^{k} \left[\left(\frac{F}{f_t}\right)^{\frac{1}{2}f_t(q-1)/F} \Gamma\left[f_t\left(\frac{1}{2}+\frac{q-1}{F}\right)\right]\right] \frac{L_1'^{-q}\, dq}{\Gamma(\frac{1}{2}F+q-1)}, \qquad \text{.....(6)} \end{aligned}$$

where Q (>1) is an arbitrary positive quantity.

Introducing as a new statistic

$$x = -F \log L_1' = -2 \log \mu, \qquad \text{.....(7)}$$

and as a new variable of integration

$$\lambda = \tfrac{1}{2} + (q-1)/F,$$

we obtain for the distribution function of x ($\psi(x)$ say)

$$\begin{aligned} \psi(x) &= \Gamma(\tfrac{1}{2}F) \prod_{t=1}^{k} \left(\frac{F}{f_t}\right)^{-\frac{1}{2}f_t} \Gamma(\tfrac{1}{2}f_t)^{-1} e^{-\frac{1}{2}x} \\ &\times \frac{1}{2\pi i} \int_{A-i\infty}^{A+i\infty} \prod_{t=1}^{k} \left\{\left(\frac{F}{f_t}\right)^{\lambda f_t} \Gamma(\lambda f_t)\right\} \frac{e^{x\lambda}}{\Gamma(F\lambda)} d\lambda, \qquad \text{.....(9)} \end{aligned}$$

where A is an arbitrary positive quantity.

Using now Binet's integral representation of $\log \Gamma$ (Whittaker & Watson, 1927, p. 249), we may write equation (9) in the form

$$\psi(x) = (\tfrac{1}{2})^{\frac{1}{2}(k-1)} e^{-E(1)} e^{-\frac{1}{2}x} \times \frac{1}{2\pi i} \int_{A-i\infty}^{A+i\infty} \lambda^{-\frac{1}{2}(k-1)} e^{x\lambda} e^{E(\lambda)} d\lambda, \qquad \text{.....(10)}$$

where
$$E(\lambda) = \int_0^\infty \left(\frac{1}{2} - \frac{1}{\tau} + \frac{1}{e^\tau - 1}\right) \frac{1}{\tau} \left[\sum_{t=1}^{k} e^{-\tau f_t \lambda} - e^{-\tau F \lambda}\right] d\tau. \qquad \text{.....(11)}$$

Introducing $$g(\tau) = \left(\frac{1}{2} - \frac{1}{\tau} + \frac{1}{e^{\tau}-1}\right)\frac{1}{\tau}, \qquad \text{.....(12)}$$

we see that $g(\tau)$ has continuous derivatives of any order for $0 \leqslant \tau \leqslant \infty$. We may therefore transform the integral (11) by integration by parts, differentiating $g(\tau)$ and integrating the exponential functions. We obtain

$$E(\lambda) = \frac{1}{12\lambda}\left[\sum_{t=1}^{k}\left(\frac{1}{f_t}\right) - \frac{1}{F}\right] - \frac{1}{360\lambda^3}\left[\sum_{t=1}^{k}\left(\frac{1}{f_t^3}\right) - \frac{1}{F^3}\right] + \frac{1}{\lambda^3}\int_0^{\infty} \frac{d^3 g(\tau)}{d\tau^3}\left[\sum_{t=1}^{k}\left(\frac{e^{-\tau\lambda f_t}}{f_t^3}\right) - \frac{e^{-\tau\lambda F}}{F^3}\right] d\tau. \qquad \text{.....(13)}$$

We now approximate to $E(\lambda)$, and therefore to $\psi(x)$, by ignoring the last summand in equation (13), and write

$$E(\lambda) \cong \frac{c_1}{12\lambda} - \frac{c_3}{360\lambda^3}, \qquad \text{.....(14)}$$

where $$c_1 = \sum_{t=1}^{k}\left(\frac{1}{f_t}\right) - \frac{1}{F} \quad \text{and} \quad c_3 = \sum_{t=1}^{k}\left(\frac{1}{f_t^3}\right) - \frac{1}{F^3}. \qquad \text{.....(15)}$$

It can be shown that this approximation is sufficient for all practical purposes provided

$$f_t \geqslant 2, \quad t = 1, 2, \ldots, k.$$

Substituting (14) in (10), expanding $e^{E(\lambda)}$ and integrating the single terms we obtain*

$$\psi(x) \cong 2^{-\frac{1}{2}(k-1)} e^{-\frac{1}{24}c_1 + \frac{1}{5760}c_3} \times \sum_{i=0}^{\infty} \alpha_i 2^{-i} \Gamma\left(\frac{k-1}{2} + i\right)^{-1} x^{\frac{1}{2}(k-3)+i} e^{-\frac{1}{2}x}, \qquad \text{.....(16)}$$

where the α_i are the coefficients of the expansion

$$e^{\frac{1}{24}c_1 t - \frac{1}{5760}c_3 t^3} \qquad \text{.....(17)}$$

in ascending powers of t.

From (16) it is obvious that (to the degree of accuracy considered) the distribution of x is a weighted sum of χ^2 distributions with degrees of freedom ranging between $k-1$ and ∞. We now denote by $P_j(X)$ the probability integral of χ^2 based on j degrees of freedom, i.e. we introduce

$$P_j(X) = \Gamma\left(\frac{j}{2}\right)^{-1} 2^{-\frac{1}{2}j} \int_X^{\infty} x^{\frac{1}{2}(j-2)} e^{-\frac{1}{2}x}\, dx. \qquad \text{.....(18)}$$

We further denote by $P(X)$ the probability integral of our variate x defined in (7), viz.

$$P(X) = \int_X^{\infty} \psi(x)\, dx. \qquad \text{.....(19)}$$

* We make use of the well-known integral representation of $1/\Gamma(z)$, viz.

$$\{\Gamma(z)\}^{-1} = \frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} e^{\rho}\rho^{-z}\, d\rho.$$

From equation (16) we obtain by integration

$$P(X) = \sum_{i=0}^{\infty} \alpha_i P_{k-1+2i}(X) \times \left(\sum_{i=0}^{\infty} \alpha_i \right)^{-1}, \qquad \text{.....(20)}$$

where the α_i are the coefficients of the expansion of

$$e^{\frac{1}{2}c_1 t - \frac{1}{4}c_2 t^2} \qquad \text{.....(21)}$$

in ascending powers of t.

3. Tabulation of percentage points

Equation (20) provides a means of calculating tables of the probability integral $P(X)$ (or its 5 % and 1 % points). For the quantities $P_j(X)$ are given by Elderton's tables of the probability integral of χ^2, while the coefficients α_i are readily obtained from the expansion of (21).

For practical purposes tables of the 5 % and 1 % points could be prepared. These percentage points would depend on three quantities, viz.

$$k, \quad c_1 = \sum_{t=1}^{k} \left(\frac{1}{f_t} \right) - \frac{1}{F} \quad \text{and} \quad c_3 = \sum_{t=1}^{k} \left(\frac{1}{f_t^3} \right) - \frac{1}{F^3}.$$

The effect of c_3 is small, and it would be convenient to make k and c_1 the respective row and column headings of two-way tables of percentage points, and to prepare such tables for two or three selected values of c_3. It is hoped to prepare such tables shortly.

4. Comparison with U. S. Nair's expansion

It would lead us too far afield if we gave here a complete mathematical proof of the accuracy of the approximation (20). It is, however, of interest to check the accuracy in a few cases numerically. U. S. Nair's expansion mentioned above will be used for this check. The most stringent test of the accuracy of equation (20) is established by choosing the f_t small and k large. Numerical results have been obtained from U. S. Nair's expansion (2) in the case $f_t = f = 2$; $k = 10$. The result of the test is given below.

Lower percentage points of $L_1' = e^{-X/F}$

	5% point	1% point
(*a*) Bartlett's approximation	0·367	0·277
(*c*) U. S. Nair's expansion	0·375	0·288
(*d*) Equation (20)	0·378	0·291

The agreement between U. S. Nair's expansion (*c*) and equation (20), (*d*) is satisfactory in this case where the approximation would be expected to be worst. For comparison, Bartlett's approximation (*a*) is also shown.

5. The relation between the special case $f_t = f$, $t = 1, 2, \ldots, k$ and the general case

P. P. N. Nayer has considered the general case, and has provided some evidence for believing that this case can be reduced to the special case provided that the f_t are not too small and not too dissimilar in value. He has suggested using the mean of the f_t as a substitute for the common value f. It is easy to see from the approximation (20) that there is some truth in Nayer's conjecture. However, it is not correct to use the arithmetic mean. The correct value is given by

$$f = \left(k - \frac{1}{k}\right)\left[\sum_{t=1}^{k}\left(\frac{1}{f_t}\right) - \frac{1}{F}\right]^{-1}, \qquad \ldots\ldots(22)$$

and is approximately equal to the harmonic mean of the f_t. If all $f_t \geqslant 4$, the general case of unequal f_t can *always* be reduced to the special case $f_t = f$, no matter how dissimilar the f_t. For if in equation (21) we replace c_3 by $\frac{1}{16}c_1$, and consider the function

$$e^{\frac{1}{2}c_1(t - \frac{1}{16}t^2)},$$

we find that the coefficients of this function when expanded in ascending powers of t will be approximations of sufficient accuracy to the coefficients α_i in (20). The probability integral of X is therefore determined by the quantities k and c_1, so that the identity of the general case and the special case is obvious, provided f is defined by (22).

6. Some remarks on Bartlett's approximation

Bartlett (1937) has given an approximation to the distribution of

$$-2\log\mu = x.$$

He suggests as an approximate test that we enter the table of χ^2 for $k-1$ degrees of freedom with the statistic

$$3x(k-1)/c_1,$$

where c_1 is given by (15).

It can be shown that this approximation is equivalent to equation (20) provided $\frac{1}{6}c_1$ is small, so that higher order terms in the expansion (20) may be ignored. For large or moderate values of $\frac{1}{6}c_1$, however, discrepancies may occur even if all mean squares are based on moderate or large numbers of degrees of freedom. We shall confine ourselves here to demonstrating this with the help of a single example, viz. $f = 5$ and $k = 30$. While for values of k of this order U. S. Nair's expansion is very complicated, equation (20) yields results which are accurate to 3 figures. Below are given the probabilities of exceeding Bartlett's 5 % and 1 % values; they are

	5 % level	1 % level
True $P(X)$	0·047	0·0081

Thus Bartlett's approximation has an error of 6 % and 19 % respectively.

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THE SIMULTANEOUS DISTRIBUTION IN SAMPLES OF MEAN AND STANDARD DEVIATION, AND OF MEAN AND VARIANCE

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In this study I propose to give the application of "the conception of the probability of passage" to the solution of the rather difficult general problem mentioned above. From this single example it is possible to deduce that the introduction of "a conception of the probability of passage" into mathematical statistics would at least make the solution of a range of difficult problems considerably easier.

Let us assume that the class symbol of the statistical element is a continuous two-dimensional variable x, y, defined in the region Ω, and that the corresponding density of the probability of passage from the class x_1, y_1 into the class x, y expressed by the symbol $p_t(x_1, y_1; x, y)$, depends, not only on the variables x, y but also on the discontinuous variable t, which is the number of operations executed on the statistical element. By operation we shall mean in our case the selection of a statistical element from the fundamental universe.

Let the function $p_t(x_1, y_1; x, y)$ satisfy the following fundamental relationship:

$$\int_\Omega \int p_t(x_1, y_1; x, y)\, dx\, dy = 1. \tag{1}$$

A further relationship, which will be used, concerns the calculations of the continuous two-dimensional probability distribution $F_{t+1}(x, y)$, corresponding to the number of operations $t+1$, from the distribution $F(x_1, y_1)$ by means of $p_t(x_1, y_1; x, y)$:

$$F_{t+1}(x, y) = \int_\Omega \int F(x_1, y_1)\, p_t(x_1, y_1; x, y)\, dx_1 dy_1. \tag{2}$$

The problem of the simultaneous distribution of the mean and standard deviation of samples in the case in which the fundamental distribution is given quite generally by a function $f(x)$, has, so far as I know, occupied the attention of only one man; this was A. T. Craig (1932) in his study: "The simultaneous distribution of mean and standard deviation in small samples." He introduces the solution only for samples of a very small number of items, $n = 2, 3, 4$.

The use of the conception of the probability of passage enables us to demonstrate successively the solution of this problem for samples with an increasing number of items. The method used gives us at the same time a solution in a very easy manner, and especially clear, if used with a graphical illustration.

I

Let us use $\bar{x}$ for the mean of a random sample of the symbols $x_1\, x_2 \ldots x_t$ taken from the continuous one-dimensional universe of density $f(x)$, where we have

$$\bar{x} = \frac{1}{t}\sum_1^t x_i, \tag{3}$$

and for the standard deviation s, which, in accordance with the definition of the standard deviation, is given by

$$s^2 = \frac{1}{t}\sum_1^t (x_i - \bar{x})^2. \tag{3·1}$$

By an extension of this sample of size t to the size $t+2$, we get a sample, the mean of which is

$$\bar{X} = \frac{1}{t+2}\sum_1^{t+2} x_i, \tag{3·2}$$

and the standard deviation S is given by

$$S^2 = \frac{1}{t+2}\sum_1^{t+2} (x_i - \bar{X})^2. \tag{3·3}$$

The elementary probability of passage $p_t(\bar{x}, s;\, \bar{X}, S)$ from the sample with $\bar{x}$ as mean and standard deviation s to the sample of mean $\bar{X}$ and standard deviation S equals the probability of the appearance of the values x_{t+1} and x_{t+2}; thus

$$f(x_{t+1})\, dx_{t+1} f(x_{t+2})\, dx_{t+2} = p_t(\bar{x}, s;\, \bar{X}, S)\, d\bar{X}\, dS. \tag{4}$$

In this expression it is necessary to substitute for the variables x_{t+1}, x_{t+2} in terms of the variables $\bar{X}$, S; the values $\bar{x}$, s in substituting being taken as constants.

From the expressions (3) and (3·2) we obtain, first of all, the following relationship:

$$x_{t+1} + x_{t+2} = \bar{X}(t+2) - \bar{x}t. \tag{5}$$

A further relationship which we obtain from equations (3·1) and (3·3) is

$$\begin{aligned}(t+2)\, S^2 &= \sum_1^t (x_i - \bar{X})^2 + (x_{t+1} - \bar{X})^2 + (x_{t+2} - \bar{X})^2 \\ &= \sum_1^t x_i^2 - 2t\bar{X}\bar{x} + (t+2)\, \bar{X}^2 + x_{t+1}^2 + x_{t+2}^2 - 2(x_{t+1} + x_{t+2})\, \bar{X}.\end{aligned}$$

If we now use the relationship

$$ts^2 = \sum_1^t x_i^2 - t\bar{x}^2,$$

we obtain another equation, which is necessary for the determination of the values x_{t+1} and x_{t+2}, i.e.

$$x_{t+1}^2 + x_{t+2}^2 = (t+2)\,(S^2 + \bar{X}^2) - t(s^2 + \bar{x}^2). \tag{5·1}$$

From equations (5) and (5·1) it then further follows that

$$\left.\begin{aligned} x_{t+1} &= \frac{t+2}{2}\bar{X} - \frac{t}{2}\bar{x} \pm \tfrac{1}{2}\sqrt{\{2[(t+2)S^2 - ts^2] - t(t+2)(\bar{X}-\bar{x})^2\}} = \frac{t+2}{2}\bar{X} - \frac{t}{2}\bar{x} \pm \tfrac{1}{2}\alpha, \\ x_{t+2} &= \frac{t+2}{2}\bar{X} - \frac{t}{2}\bar{x} \mp \tfrac{1}{2}\sqrt{\{2[(t+2)S^2 - ts^2] - t(t+2)(\bar{X}-\bar{x})^2\}} = \frac{t+2}{2}\bar{X} - \frac{t}{2}\bar{x} \mp \tfrac{1}{2}\alpha, \end{aligned}\right\} \tag{5·2}$$

where, for the sake of brevity, we introduce the symbol

$$\alpha = \sqrt{\{2[(t+2)S^2 - ts^2] - t(t+2)(\bar{X}-\bar{x})^2\}}.$$

For effecting the substitution in expression (4) it is also necessary to know the corresponding determinant of the substitution

$$\left|\frac{D(x_{t+1}, x_{t+2})}{D(\bar{X}, S)}\right| = \frac{(t+2)^2 S}{2\sqrt{\{2[(t+2)S^2 - ts^2] - t(t+2)(\bar{X}-\bar{x})^2\}}}.$$

Referring to the two different values for each of the symbols x_{t+1} and x_{t+2} we then get the expression for the density of probability of passage $p_t(\bar{x}, s;\ \bar{X}, S)$ in the form

$$f\left(\frac{t+2}{2}\bar{X} - \frac{t}{2}\bar{x} + \tfrac{1}{2}\alpha\right) f\left(\frac{t+2}{2}\bar{X} - \frac{t}{2}\bar{x} - \tfrac{1}{2}\alpha\right) \frac{2(t+2)^2 S}{\alpha}. \tag{6}$$

Let $F_t(\bar{x}, s)$ be the density of the simultaneous distribution of mean $\bar{x}$ and standard deviation s in the random sample of size t.

The density of the distribution of mean $\bar{X}$ and standard deviation S in the random sample of size $t+2$ is obtained by application of the fundamental relationship (2), and for its value we get the following expression:

$$\begin{aligned} F_{t+2}(\bar{X}, S) &= \int_\Omega \int F_t(\bar{x}, s)\, p_t(\bar{x}, s;\ \bar{X}, S)\, d\bar{x}\, ds \\ &= 2S(t+2)^2 \int_\Omega \int F_t(\bar{x}, s) f\left(\frac{t+2}{2}\bar{X} - \frac{t}{2}\bar{x} + \tfrac{1}{2}\alpha\right) f\left(\frac{t+2}{2}\bar{X} - \frac{t}{2}\bar{x} - \tfrac{1}{2}\alpha\right) \frac{d\bar{x}\, ds}{\alpha}. \end{aligned} \tag{7}$$

For the complete solution of this recurrence relationship, besides the limits of integration (which we shall consider later on), we need to know the initial values of the function $F_t(\bar{X}, S)$, i.e.

$$\left.\begin{aligned} F_1(\bar{X}, S) &= f(\bar{X}), \\ F_2(\bar{X}, S) &= p_0(\bar{x}, s;\ \bar{X}, S) = 4f(\bar{X}+S) f(\bar{X}-S). \end{aligned}\right\} \tag{7·1}$$

Let us supplement these values with the following function $F_3(\bar{X}, S)$ for which only one integration is necessary:

$$\begin{aligned} F_3(\bar{X}, S) = 18S \int & f(\bar{x}) f[\tfrac{3}{2}\bar{X} - \tfrac{1}{2}\bar{x} + \tfrac{1}{2}\sqrt{\{6S^2 - 3(\bar{X}-\bar{x})^2\}}] \\ & \times f[\tfrac{3}{2}\bar{X} - \tfrac{1}{2}\bar{x} - \tfrac{1}{2}\sqrt{\{6S^2 - 3(\bar{X}-\bar{x})^2\}}] \frac{d\bar{x}}{\sqrt{\{6S^2 - 3(\bar{X}-\bar{x})^2\}}}, \end{aligned}$$

the limits of integration will be deduced later on in this study.

In order to examine the limits of integration,

(1) *Let us first of all assume that the density of the fundamental universe is expressed by the function $f(x)$, defined between the limits $\pm\infty$.*

In this case we set only one condition on the values x_{t+1}, x_{t+2}, that is that they must be real. The condition corresponding to this is expressed by the inequality

$$2(t+2)\,S^2 - 2ts^2 - t(t+2)\,(\bar{X}-\bar{x})^2 \geqslant 0.$$

If we consider $\bar{X}$, S as constants and $\bar{x}$, s as variables and we use rectangular co-ordinates with axes $\bar{x}$, s, these variables are limited to the region given by that half of the ellipse E_0:

$$\frac{t+2}{2}(\bar{X}-\bar{x})^2 + s^2 = \frac{t+2}{t}S^2,$$

which lies above the axis $\bar{x}$. According to the choice of the values $\bar{X}$, S, the variable $\bar{x}$ varies between the limits $\pm\infty$; the variable s then lies in the range $0, \infty$.

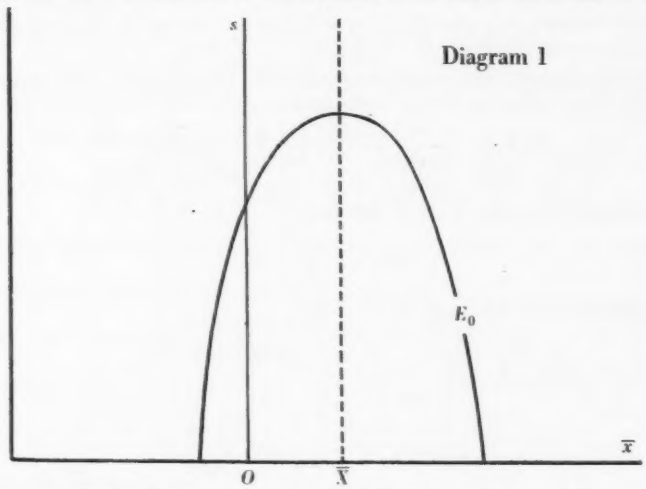


Diagram 1

The centre of the ellipse E_0 lies on the axis of $\bar{x}$ at a distance $\bar{X}$ from the origin of the co-ordinates; the semi-axes have the lengths

$$\lambda_s = S\sqrt{\frac{t+2}{t}}, \quad \lambda_{\bar{x}} = \sqrt{\frac{2}{t}}.$$

The integration of expression (7) with respect to $\bar{x}$ must therefore be carried out between the limits

$$\left\{\bar{X} - \sqrt{\left(\frac{2S^2}{t} - \frac{2s^2}{t+2}\right)},\ \bar{X} + \sqrt{\left(\frac{2S^2}{t} - \frac{2s^2}{t+2}\right)}\right\},$$

and then with respect to s between the limits

$$\left(0, S\sqrt{\frac{t+2}{t}}\right).$$

In Diagram 1 the surface of integration is shown for the particular values $\bar{X} = 1$, $S = 1{\cdot}5$, $t = 4$.

The ellipse E_0 intersects the $\bar{x}$ axis in the points $\bar{X} \pm S\sqrt{\frac{2}{t}}$, the semi-major axis λ_s being always larger than $\lambda_{\bar{x}}$, their ratio

$$\frac{\lambda_s}{\lambda_{\bar{x}}} = \sqrt{\frac{t+2}{2}}$$

depending only on the size of the sample.

If calculation of the value of $F_3(\bar{X}, S)$, is involved, i.e. if $t = 1$, the ellipse E_0 reduces to that part of the $\bar{x}$ axis with $\bar{X} \pm S\sqrt{2}$ as end-points. The corresponding integration need only be carried out with respect to $\bar{x}$, and that between the limits

$$\bar{X} \pm S\sqrt{2}.$$

Formula (7) can be used, not only for the successive calculation of the simultaneous distribution of the mean and standard deviation for an increasing size of random samples, but also for the verification of the given distribution values introduced for the arbitrary t.

Thus, for example, it is possible to check the correctness of the expression

$$F_t(\bar{X}, S) = \frac{\sqrt{t}}{c\sqrt{(2\pi)}} e^{-\frac{t\bar{X}^2}{2c^2}} 2\left(\frac{t}{2c^2}\right)^{\frac{t-1}{2}} \frac{S^{t-2}}{\Gamma\left(\frac{t-1}{2}\right)} e^{-\frac{tS^2}{2c^2}}$$

which corresponds to the normal universe

$$f(x) = \frac{1}{c\sqrt{2\pi}} e^{-\frac{x^2}{2c^2}}.$$

By application of formula (7) we obtain

$$F_{t+2}(\bar{X}, S) = \left(\frac{t}{2}\right)^{\frac{t-1}{2}} \left(\frac{t+2}{\pi}\right)^{\frac{1}{2}} \frac{S}{c^{t+2}\Gamma\left(\frac{t-1}{2}\right)} e^{-\frac{(t+2)(\bar{X}^2+S^2)}{2c^2}}$$

$$\times \int_0^S \sqrt{\frac{t+2}{t}}\, s^{t-2} ds \int_{\bar{X}-\sqrt{\left(\frac{S^2}{t}-\frac{s^2}{t+2}\right)}}^{\bar{X}+\sqrt{\left(\frac{S^2}{t}-\frac{s^2}{t+2}\right)}} \frac{d\bar{x}}{\sqrt{\left(\frac{S^2}{t}-\frac{s^2}{t+2}-\frac{1}{2}(\bar{X}-\bar{x})^2\right)}}$$

$$= \left(\frac{t}{2}\right)^{\frac{t-1}{2}} \left(\frac{t+2}{\pi}\right)^{\frac{1}{2}} \frac{S}{c^{t+2}\Gamma\left(\frac{t-1}{2}\right)} e^{-\frac{(t+2)(\bar{X}^2+S^2)}{2c^2}}$$

$$\times \int_0^S \sqrt{\frac{t+2}{t}}\, s^{t-2} ds \left[\frac{1}{\sqrt{2}} \arcsin \frac{-\frac{1}{2}\bar{x}+\bar{X}}{\sqrt{\left(\frac{S^2}{t}-\frac{s^2}{t+2}\right)}}\right]_{\bar{X}+\sqrt{\left(\frac{S^2}{t}-\frac{s^2}{t+2}\right)}}^{\bar{X}-\sqrt{\left(\frac{S^2}{t}-\frac{s^2}{t+2}\right)}}$$

$$= \frac{\sqrt{(t+2)}}{c\sqrt{(2\pi)}} e^{-\frac{(t+2)\bar{X}^2}{2c^2}} 2\left(\frac{t+2}{2c^2}\right)^{\frac{t+1}{2}} \frac{S^t}{\Gamma\left(\frac{t+1}{2}\right)} e^{-\frac{(t+2)S^2}{2c^2}}.$$

As can be seen, the function $F_t(\bar{X}, S)$ satisfies the fundamental recurrence relationship (7).

(2) *Let the fundamental universe be defined by a continuous function defined in the range* $0, \infty$.

Under this assumption the values x_{t+1}, x_{t+2}, must satisfy, apart from the condition of their being real, another condition expressed by the inequality

$$\frac{t+2}{2}\bar{X} - \frac{t}{2}\bar{x} - \tfrac{1}{2}\sqrt{\{2(t+2)S^2 - 2ts^2 - t(t+2)(\bar{X}-\bar{x})^2\}} \geqslant 0;$$

the mean of the sample $\bar{x}$ cannot then assume negative values, which gives

$$\bar{x} \geqslant 0.$$

The corresponding values of the variables $\bar{x}$, s, satisfying the first inequality lie outside the ellipse E_1

$$\frac{t(t+1)}{2}\bar{x}^2 - t(t+2)\bar{X}\bar{x} + \frac{t}{2}s^2 + \frac{(t+1)(t+2)}{2}\bar{X}^2 - \frac{t+2}{2}S^2 = 0.$$

The centre of the ellipse is situated on the $\bar{x}$ axis at a distance $\bar{X}(t+2)/(t+1)$ from the origin, the semi-axes have the lengths

$$\lambda'_s = \sqrt{\frac{t+2}{t}}\sqrt{\left(S^2 - \frac{\bar{X}^2}{t+1}\right)}; \quad \lambda'_{\bar{x}} = \sqrt{\frac{t+2}{t(t+1)}}\sqrt{\left(S^2 - \frac{\bar{X}^2}{t+1}\right)}.$$

From the quotient $\dfrac{\lambda'_s}{\lambda'_{\bar{x}}} = \sqrt{(t+1)}$ it follows that the semi-axis λ'_s is longer than $\lambda'_{\bar{x}}$. The ellipse E_1 intersects the $\bar{x}$ axis at the points

$$\left\{\bar{X}\frac{t+2}{t+1} \pm \sqrt{\frac{t+2}{t(t+1)}}\sqrt{\left(S^2 - \frac{\bar{X}^2}{t+1}\right)}\right\}.$$

The ellipse E_1 lies inside the ellipse E_0 and touches it at the point $\bar{x}_1$, s_1, where

$$\bar{x}_1 = \bar{X}\frac{t+2}{t}, \quad s_1 = \sqrt{\frac{t+2}{t}}\sqrt{\left(S^2 - \frac{2\bar{X}^2}{t}\right)}$$

as long as

$$S \geqslant \bar{X}\sqrt{\frac{2}{t}}.$$

From the expression for the lengths of the semi-axes it follows that E_1 is real only as long as the condition

$$S \geqslant \frac{\bar{X}}{\sqrt{(t+1)}}$$

is satisfied.

Considering the condition that the mean $\bar{x}$ cannot attain negative values, we obtain the inequality

$$S \leqslant \bar{X}\sqrt{(t+1)}.$$

The system of the ellipses E_1 is therefore real for those points with the parameters $\bar{X}$, S as co-ordinates, which lie between the straight lines

$$S = \frac{\bar{X}}{\sqrt{(t+1)}}; \quad S = \bar{X}\sqrt{(t+1)}$$

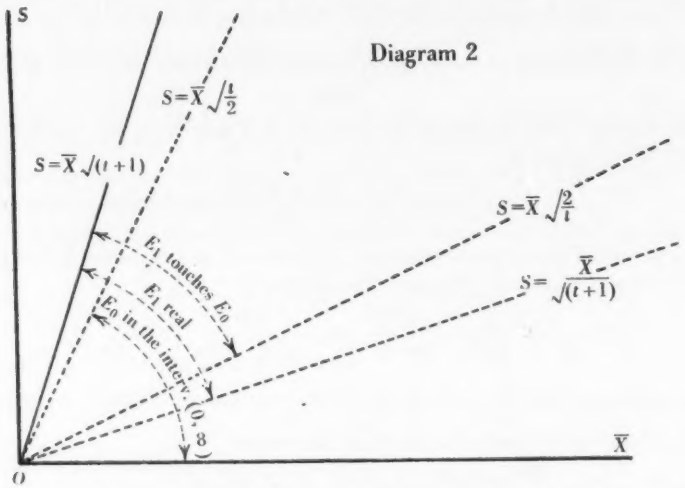
and that for $\bar{X} \geqslant 0$, $S \geqslant 0$.

When carrying out the integration indicated in formula (7), it is also necessary to distinguish between the cases when the ellipse E_0 intersects the negative part of the $\bar{x}$ axis and when it does not, i.e. if

$$S \leqslant \bar{X}\sqrt{\frac{t}{2}} \quad \text{or} \quad S \geqslant \bar{X}\sqrt{\frac{t}{2}}.$$

In Diagram 2, the corresponding regions of the values $\bar{X}$, S are shown for $t = 8$.

Diagram 2



The surface of integration is that part of the plane $\bar{x}$, s in the first quadrant contained between the ellipses E_0 and E_1.

The process of integration is then as follows:

(*a*) Let S satisfy the inequality

$$0 \leqslant S \leqslant \frac{\bar{X}}{\sqrt{(t+1)}}.$$

In this case only the ellipse E_0 is real. It is therefore necessary to carry out the integration with respect to $\bar{x}$ in between the limits $\bar{x} \pm \sqrt{\left(\frac{2S^2}{t} - \frac{2s^2}{t+2}\right)}$ and with respect to s between the limits 0, $S\sqrt{\frac{t+2}{t}}$.

In order to illustrate the theory, formula (7) will be applied to find the analytical expression of the correlation surface $F_t(\bar{X}, S)$ for samples from the fundamental universe $f(x) = e^{-x}; \quad 0 \leqslant x \leqslant \infty$

over the region in the plane $\bar{X}$, S bounded by the inequality mentioned above. For $t = 2$ and 3 the results are given in A. T. Craig's paper, namely

$$F_2(\bar{X}, S) = 4e^{-2\bar{X}}; \quad F_3(\bar{X}, S) = 6\sqrt{3}\,\pi S e^{-3\bar{X}}.$$

By application of our formula (7) we obtain

$$F_4(\overline{X}, S) = 64\pi S^2 e^{-4\overline{X}}; \quad F_5(\overline{X}, S) = 50\sqrt{5}\,\pi^2 S^3 e^{-5\overline{X}}.$$

Regarding these expressions let us assume the general solution to be

$$F_{t+2} = H_{t+2} S^t e^{-(t+2)\overline{X}}.$$

A recurrence relationship for the coefficients H_t may be easily found from the equation

$$F_{t+2}(\overline{X}, S) = 2S(t+2)^2 \int_\Omega \int H_t s^t e^{-(t+2)\overline{x}} \frac{d\overline{x}\,ds}{\alpha} = H_t 2S^t \pi e^{-(t+2)\overline{X}} \frac{(t+2)^{\frac{t+2}{2}}}{(t-1)t^{\frac{t}{2}}}$$

$$= H_{t+2} S^t e^{-(t+2)\overline{X}},$$

$$H_{t+2} = H_t 2\pi \frac{(t+2)^{\frac{t+2}{2}}}{(t-1)\,t^{\frac{t}{2}}}.$$

It is then not difficult to verify that for $t \geqslant 2$

$$H_t = \frac{2\pi^{\frac{t+1}{2}}\, t^{\frac{t}{2}}}{\Gamma\left(\frac{t-1}{2}\right)}.$$

The function $$F_t(\overline{X}, S) \frac{2\pi^{\frac{t+1}{2}}\, t^{\frac{t}{2}}}{\Gamma\left(\frac{t-1}{2}\right)} S^{t-2} e^{-t\overline{X}}, \quad t \geqslant 2,$$

represents a part of the whole distribution given by the integral

$$\int_{\overline{X}=0}^{\infty} \int_{S=0}^{\frac{\overline{X}}{\sqrt{(t+1)}}} F_t(\overline{X}, S)\, d\overline{X}\, dS = \frac{2\pi^{\frac{t-1}{2}}\, \Gamma(t)}{t^{\frac{t}{2}} (t-1)^{\frac{t+1}{2}}\, \Gamma\left(\frac{t-1}{2}\right)}, \quad t \geqslant 2,$$

that is, if

$t =$	2	3	4	5
	100 %	c. 60 %	c. 30 %	c. 13 %

of the whole distribution.

(*b*) If S satisfies the inequality

$$\frac{\overline{X}}{\sqrt{(t+1)}} \leqslant S \leqslant \overline{X} \sqrt{\left(\frac{t}{2}\right)}$$

the relative position of the ellipses E_0 and E_1 is shown in Diagrams 3*a*, *b*. The integral in formula (7) is equal to the sum of the integrals between the following limits:

(α) According to $\overline{x}$:

$$\overline{X} - \sqrt{\left(\frac{2S^2}{t} - \frac{2s^2}{t+2}\right)}; \quad \overline{X}\frac{t+2}{t+1} - \sqrt{\frac{t+2}{t(t+1)}} \sqrt{\left(S^2 - s^2 \frac{t}{t+2} - \frac{\overline{X}^2}{t+1}\right)}.$$

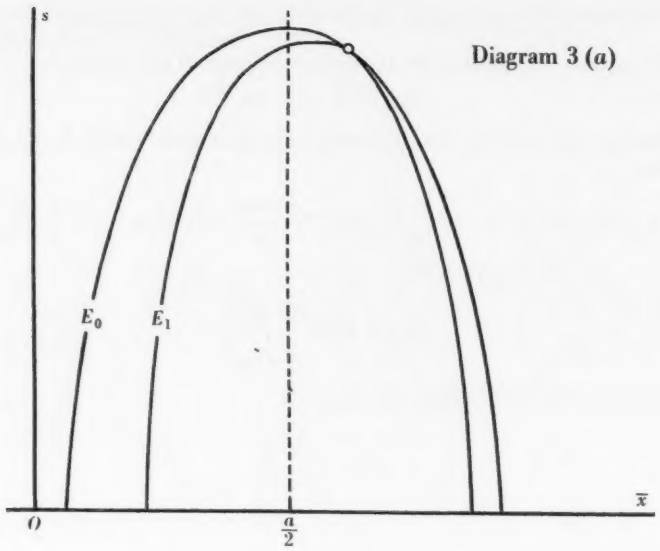


Diagram 3 (*a*)

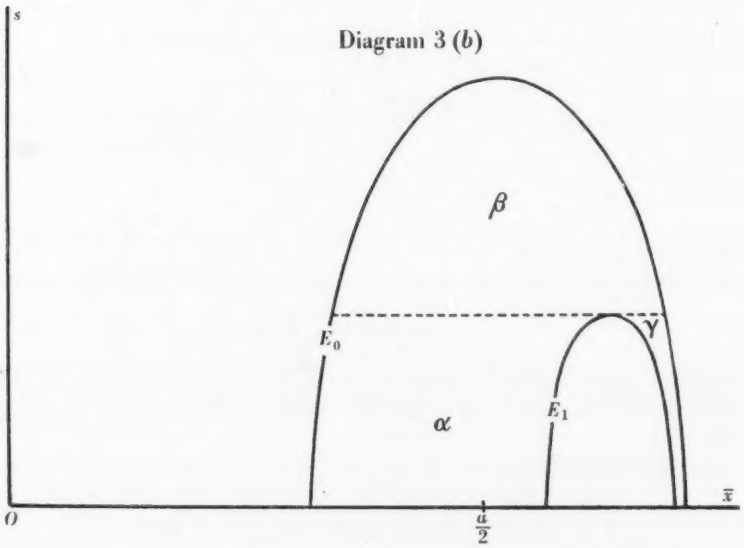


Diagram 3 (*b*)

According to s:

$$0; \quad \sqrt{\frac{t+2}{t}}\sqrt{\left(S^2-\frac{\bar{X}^2}{t+1}\right)}.$$

(β) According to $\bar{x}$:

$$\bar{X}-\sqrt{\left(\frac{2S^2}{t}-\frac{2s^2}{t+2}\right)}; \quad \bar{X}+\sqrt{\left(\frac{2S^2}{t}-\frac{2s^2}{t+2}\right)}.$$

According to s:

$$\sqrt{\frac{t+2}{t}}\sqrt{\left(S^2\frac{\bar{X}^2}{t+1}\right)}; \quad S\sqrt{\frac{t+2}{t}}.$$

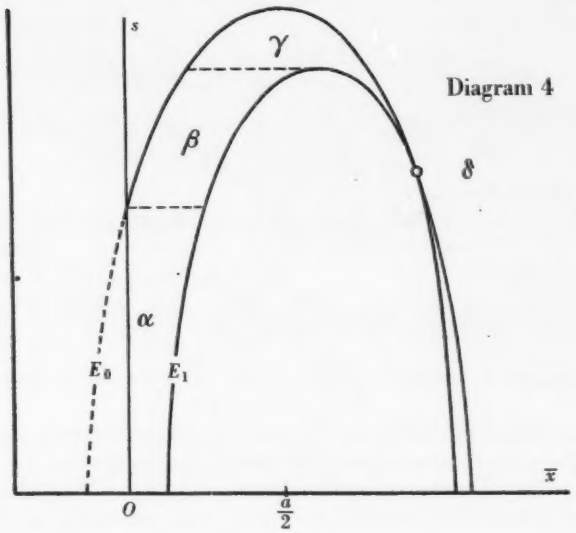


Diagram 4

(γ) According to $\bar{x}$:

$$\bar{X}\frac{t+2}{t+1}+\sqrt{\frac{t+2}{t(t+1)}}\sqrt{\left(S^2-s^2\frac{t}{t+2}-\frac{\bar{X}^2}{t+1}\right)}; \quad \bar{X}+\sqrt{\left(\frac{2S^2}{t}-\frac{2s^2}{t+2}\right)}.$$

According to s:

$$0; \quad \sqrt{\frac{t+2}{t}}\sqrt{\left(S^2-\frac{\bar{X}^2}{t+1}\right)}.$$

(c) Finally if $\bar{X}\sqrt{\frac{t}{2}} \leqslant S \leqslant \bar{X}\sqrt{(t+1)}$, part of the ellipse E_0 falls to the left of the s axis, E_1 always being to the right of the s axis, the surface of integration of the values $\bar{x}$, s is made up of two parts, as can be seen in Diagram 4.

For the purpose of integration it is necessary to divide the integration into four parts: α, β, γ, δ. The integral of the expression given in (7) is equal to the sum

of the integrals with the following limits for the value $\bar{x}$, and for s, $\bar{x}$ to be taken first of all:

(α) The limits of $\bar{x}$:

$$0; \quad \bar{X}\frac{t+2}{t+1} - \sqrt{\frac{t+2}{t(t+1)}}\sqrt{\left(S^2 - \frac{s^2 t}{t+2} - \frac{\bar{X}^2}{t+1}\right)}.$$

The limits of s: $\quad 0; \quad \sqrt{\left(\frac{t+2}{t}S^2 - \frac{t+2}{2}\bar{X}^2\right)}.$

(β) The limits of $\bar{x}$:

$$\bar{X} - \sqrt{\left(\frac{2S^2}{t} - \frac{2s^2}{t+2}\right)}; \quad \bar{X}\frac{t+2}{t+1} - \sqrt{\frac{t+2}{t(t+1)}}\sqrt{\left(S^2 - \frac{s^2 t}{t+2} - \frac{\bar{X}^2}{t+1}\right)}.$$

The limits of s:

$$\sqrt{\frac{t+2}{t}}\sqrt{\left(S^2 - \frac{t+2}{2}\bar{X}^2\right)}; \quad \sqrt{\left(S^2 - \frac{\bar{X}^2}{t+1}\right)}\sqrt{\frac{t+2}{t}}.$$

(γ) The limits of $\bar{x}$:

$$\bar{X} - \sqrt{\left(\frac{2S^2}{t} - \frac{2s^2}{t+2}\right)}; \quad \bar{X} + \sqrt{\left(\frac{2S^2}{t} - \frac{2s^2}{t+2}\right)}.$$

The limits of s:

$$\sqrt{\frac{t+2}{t}}\sqrt{\left(S^2 - \frac{\bar{X}^2}{t+1}\right)}; \quad S\sqrt{\frac{t+2}{t}}.$$

(δ) The limits of $\bar{x}$:

$$\bar{X}\frac{t+2}{t+1} + \sqrt{\frac{t+2}{t(t+1)}}\sqrt{\left(S^2 - \frac{s^2 t}{t+2} - \frac{\bar{X}^2}{t+1}\right)}; \quad \bar{X} + \sqrt{\left(\frac{2S^2}{t} - \frac{2s^2}{t+2}\right)}.$$

The limits of s: $\quad 0; \quad \sqrt{\frac{t+2}{t}}\sqrt{\left(S^2 - \frac{\bar{X}^2}{t+1}\right)}.$

If the calculation of function $F_3(\bar{X}, S)$ is especially required, that means, if $t = 1$, that there is a substantial simplification in the integration. First of all it is necessary to carry out the integration only according to $\bar{x}$; besides that it is sufficient to differentiate between two cases only, according to whether the value S satisfies

$$0 \leqslant S \leqslant \frac{\bar{X}}{\sqrt{2}} \quad \text{or} \quad \frac{\bar{X}}{\sqrt{2}} \leqslant S \leqslant \bar{X}\sqrt{2},$$

since obviously the case of $\frac{\bar{X}}{\sqrt{(t+1)}} \leqslant S \leqslant \bar{X}\sqrt{\frac{t}{2}}$ drops out of consideration.

If the first of the given inequalities is valid, it is necessary to carry out the integration with respect to $\bar{x}$ between the limits:

$\bar{X} - S\sqrt{2}$; $\bar{X} + S\sqrt{2}$; in the case of the validity of the second inequality, the corresponding limits of integration are:

$$0; \quad \tfrac{3}{2}\bar{X} - \tfrac{1}{2}\sqrt{(6S^2 - 3\bar{X}^2)},$$

$$\tfrac{3}{2}\bar{X} + \tfrac{1}{2}\sqrt{(6S^2 - 3X^2)}; \quad \bar{X} + S\sqrt{2}.$$

These results agree with the results quoted in the paper by A. T. Craig already cited.

(3) *Let the fundamental universe be defined in the range* $0, a$.

The values x_{t+1}, x_{t+2}, in this case are determined by the inequality

$$0 \leqslant x_{t+1}, \quad x_{t+2} \leqslant a,$$

the mean of the sample satisfies the inequality

$$0 \leqslant \bar{x} \leqslant a.$$

From the upper limit of the values x_{t+1}, x_{t+2} it follows that

$$\frac{t+2}{2}\bar{X} - \frac{t}{2}\bar{x} + \tfrac{1}{2}\sqrt{\{2(t+2)S^2 - 2ts^2 - t(t+2)(\bar{X}-\bar{x})^2\}} \leqslant a.$$

The corresponding values of the variables $\bar{x}$, s, which satisfy this inequality, lie outside the ellipse E_2:

$$\frac{t(t+1)}{2}\bar{x}^2 + t\bar{x}(a - \overline{t+2}\,\bar{X}) + \frac{t}{2}s^2 = \frac{t+2}{t}S^2 - \frac{(t+1)(t+2)}{2}\bar{X}^2 + a(t+2)\bar{X} - a^2.$$

Its centre lies on the $\bar{x}$ axis and is at a distance from the origin equal to

$$\bar{X}\frac{t+2}{t+1} - \frac{a}{t+1},$$

this distance is always smaller than the distance of the centre of the ellipse E_0 from the origin.

The lengths of the semi-axes are

$$\lambda_x'' = \sqrt{\frac{t+2}{t(t+1)}}\sqrt{\left(S^2 - \frac{(a-\bar{X})^2}{t+1}\right)} = \frac{\lambda_x''}{\sqrt{(t+1)}}.$$

The ellipse E_2 lies inside the ellipse E_0, and touches it at the point where

$$\bar{x}_2 = \bar{X}\frac{t+2}{t} - \frac{2a}{t}; \quad s_2 = \sqrt{\frac{t+2}{t}}\sqrt{\left(S^2 - \frac{2}{t}(a-\bar{X})^2\right)}$$

as long as

$$S \geqslant (a-\bar{X})\sqrt{\frac{2}{t}}.$$

If the ellipse E_2 is to be real, then we must have

$$S \geqslant \frac{a-\bar{X}}{\sqrt{(t+1)}}.$$

From the upper limit a of the mean $\bar{x}$ of the sample it follows that

$$\bar{X}\frac{t+2}{t+1} - \frac{a}{t+1} + \sqrt{\frac{t+2}{t(t+1)}}\sqrt{\left(S^2 - \frac{(a-\bar{X}^2)}{t+1}\right)} \leqslant a.$$

This inequality leads to the relation

$$S \leqslant (a-\bar{X})\sqrt{(t+1)}.$$

Finally, it is necessary to consider the condition that the ellipses E_1 and E_2 must not intersect, in the limiting case they touch at the point

$$\bar{x} = \bar{X}\frac{t+2}{t} - \frac{a}{t}; \quad s = \sqrt{\left(\frac{t+2}{t}S^2 + 2\bar{X}\frac{t+2}{t^2}(a-\bar{X}) - a^2\frac{t+1}{t^2}\right)} = 0.$$

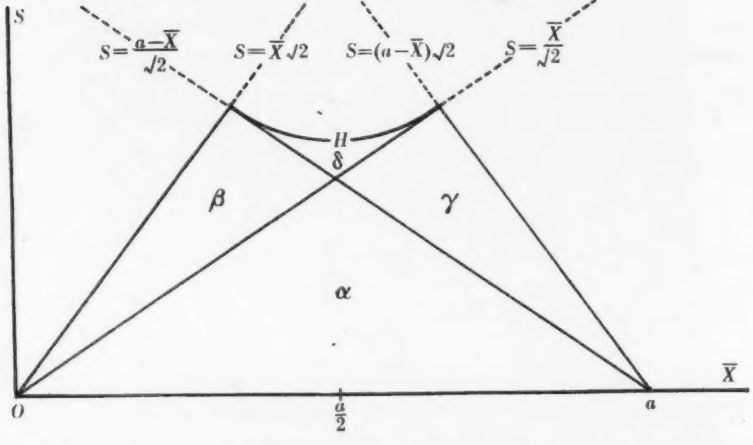
From this condition there follows the limitation of the permissible values of $\bar{X}$, S by the hyperbola H

$$S^2 + 2\bar{X}\frac{a\bar{X}}{t} - a^2\frac{t+1}{t(t+2)} = 0.$$

For carrying out the integration in formula (7) it is necessary to know the condition for the ellipse E_0 to intersect the $\bar{x}$ axis at a distance λ from the origin, i.e.

$$S = (a - \bar{X})\sqrt{\frac{t}{2}}.$$

Diagram 5



The whole process of integration is quite clearly shown in the graphical representation of the individual segments of the permissible values $\bar{X}$, S in Diagrams 5–7, for $t = 1, 2$ and 3. It is convenient to split up the surface of the values $\bar{X}$, S, for which both the ellipses E_1 and E_2 exist, into two parts by a straight line $\bar{X} = \frac{1}{2}a$.

If $$\bar{X} < \tfrac{1}{2}a,$$

the semi-axis λ_2' of the ellipse E_1 is longer than that of the ellipse E_2.

If $$\bar{X} > \tfrac{1}{2}a,$$

the ratio of the lengths of the semi-axes is the reciprocal.

The description of the limits of integration for $\bar{x}$, s is lengthy in tabular form and is better given by means of diagrams.

In the special case, $t = 1$, the integration is carried out only according to $\bar{x}$, for the values $\bar{X}$, S (see Diagram 5) in the segment:

(α) Between the limits: $\bar{X} - S\sqrt{2}$; $\bar{X} + S\sqrt{2}$.

(β) Between the limits:

$$0; \quad \tfrac{3}{2}\bar{X} - \tfrac{1}{2}\sqrt{(6S^2 - 3\bar{X}^2)} \quad \text{and} \quad \tfrac{3}{2}\bar{X} + \tfrac{1}{2}\sqrt{(6S^2 - 3\bar{X}^2)}; \quad \bar{X} + S\sqrt{2}.$$

(γ) Between the limits:

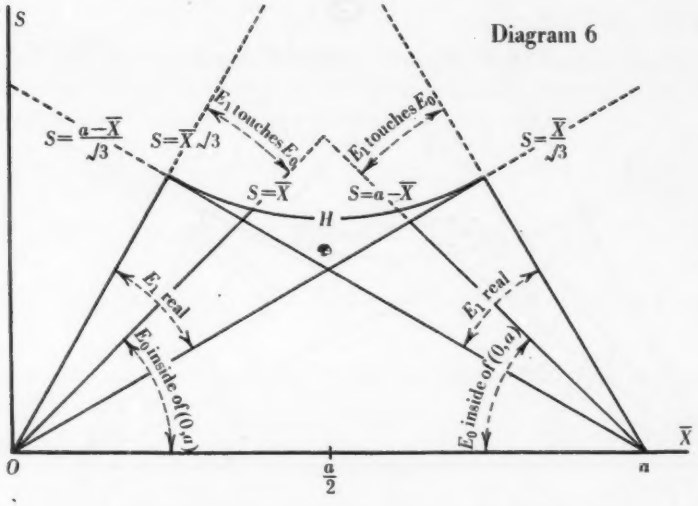
$$\bar{X} - S\sqrt{2};\quad \frac{3\bar{X}-a}{2} - \tfrac{1}{2}\sqrt{\{6S^2 - 3(a-\bar{X})^2\}} \quad \text{and} \quad \frac{3\bar{X}-a}{2} + \tfrac{1}{2}\sqrt{\{6S^2 - 3(a-\bar{X})^2\}};\quad a.$$

(δ) Between the limits:

$$0;\quad \frac{3\bar{X}-a}{2} - \tfrac{1}{2}\sqrt{\{6S^2 - 3(a\bar{X})^2\}} \quad \text{and} \quad \frac{3\bar{X}-a}{2} + \tfrac{1}{2}\sqrt{\{6S^2 - 3(a\bar{X})^2\}};$$

$$\frac{3\bar{X}}{2} - \tfrac{1}{2}\sqrt{\{6S^2 - 3\bar{X}^2\}} \quad \text{and} \quad \tfrac{3}{2}\bar{X} + \tfrac{1}{2}\sqrt{\{6S^2 - 3\bar{X}^2\}};\quad a.$$

Diagram 6



As a simple illustration let us find the correlation surface $F_t(\bar{X}, S)$ of samples of t items drawn from the distribution

$$f(x) = \frac{1}{a};\quad 0 \leqslant x \leqslant a$$

over the region bounded by the straight lines

$$S = 0,\quad S\frac{\bar{X}}{\sqrt{(t-1)}},\quad S\frac{a-\bar{X}}{\sqrt{(t-1)}}.$$

Using the results given by A. T. Craig,

$$F_2(\bar{X}, S) = \frac{4}{a^2};\quad F_3(\bar{X}, S) = \frac{6\sqrt{3}\,\pi S}{a^3},$$

we find, by our formula (7),

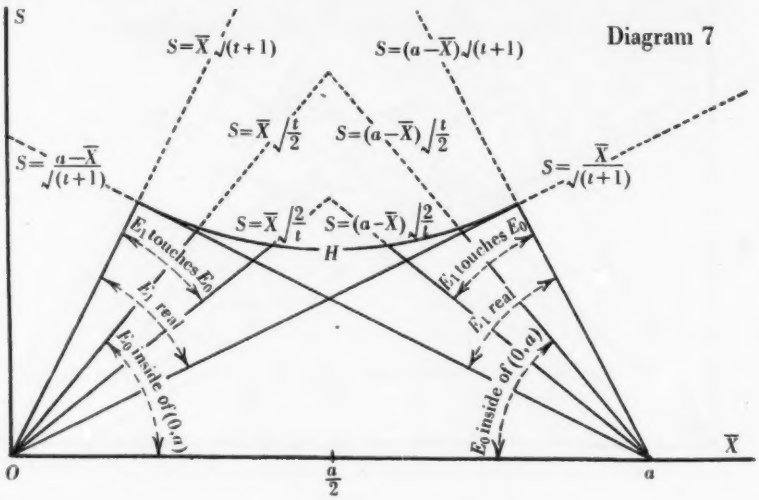
$$F_4(\bar{X}, S) = \frac{64\pi S^2}{a^4};\quad F_5(\bar{X}, S) = \frac{50\sqrt{5}\,\pi^2 S^3}{a^5},$$

and then by similar reasoning as in example on page 262 we obtain

$$F_t(\overline{X}, S) = \frac{2\pi^{\frac{t-1}{2}} t^{\frac{t}{2}}}{a^t \Gamma\left(\frac{t-1}{2}\right)} S^{t-2}.$$

This function is independent of the variable $\overline{X}$.

As to statistical *theory*, the problem of the correlation surface $F_t(\overline{X}, S)$ is solved by our formula (7) for any fundamental population $f(x)$, but as to *application* on a special distribution $f(x)$, I have not overcome all the difficulties of



integration. Nevertheless, I feel, the approach being a new one, this study may be of interest to statisticians and I hope perhaps that some mathematician will see how to solve the problems that I have left uncompleted.

II

Let us take the density of the simultaneous distribution of a mean $\bar{x}$ and variance $s^2 = u$, when the size of the sample is t, as $G_t(\bar{x}, u)$. By an application of the same process as in Part I we get the fundamental recurrence relationship for the successive calculation of the values $G_t(\bar{x}, u)$:

$$G_{t+2}(\overline{X}, U) = (t+2)^2 \int_\Omega \int G_t(\bar{x}, u) f\left(\frac{t+2}{2}\overline{X} - \frac{t}{2}\bar{x} + \tfrac{1}{2}\alpha\right) f\left(\frac{t+2}{2}\overline{X} - \frac{t}{2}\bar{x} - \tfrac{1}{2}\alpha\right) \frac{d\bar{x}\,du}{\alpha}. \tag{8}$$

The corresponding initial values are expressed by the following relationship:

$$\left.\begin{aligned} G_1(\bar{X}, U) &= f(\bar{X}) \\ G_2(\bar{X}, U) &= \frac{2}{\sqrt{U}} f(\bar{X}+\sqrt{U}) f(\bar{X}-\sqrt{U}). \end{aligned}\right\} \tag{8·1}$$

For the calculation of the next value $G_3(\bar{X}, U)$ we need to carry out only one simple integration:

$$G_3(\bar{X}, U) = 9\int f(\bar{x}) f[\tfrac{3}{2}\bar{X} - \tfrac{1}{2}\bar{x} + \tfrac{1}{2}\sqrt{\{6U - 2u - 6(\bar{X}-\bar{x})^2\}}]$$
$$\times f[\tfrac{3}{2}\bar{X} - \tfrac{1}{2}\bar{x} - \tfrac{1}{2}\sqrt{\{6U - 2u - 6(\bar{X}-\bar{x})^2\}}] \frac{d\bar{x}}{\sqrt{\{6U - 2u - 6(\bar{X}-\bar{x})^2\}}}.$$

The limits of integration for the different ranges of the fundamental universe can be deduced by the method given in Part I. At the same time the three ellipses E_0, E_1, E_2 are replaced by the three parabolae, P_0, P_1 and P_2:

$$\frac{t+2}{2}(\bar{x}-\bar{X})^2 + u = \frac{t+2}{t} U,$$

$$\frac{t(t+1)}{2}\bar{x}^2 - t(t+2)\bar{x}\bar{X} + \frac{t}{2}u = \frac{t+2}{2}U - \frac{(t+1)(t+2)}{2}\bar{X},$$

$$\frac{t(t+1)}{2}\bar{x}^2 - t\bar{x}(a - \overline{t+2}\,\bar{X}) + \frac{t}{2}u = \frac{t+2}{2}U - \frac{(t+1)(t+2)}{2}\bar{X}^2 + a(t+2)\bar{X} - a^2$$

In conclusion I must express my thanks to Professor E. S. Pearson for advice and several useful suggestions.

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CERTAIN PROJECTIVE DEPTH AND BREADTH MEASUREMENTS OF THE FACIAL SKELETON IN MAN

By ALETTE SCHREINER, *Oslo*

1. Definitions of the measurements

In their study of the "flatness" of the facial skeleton in man T. L. Woo & G. M. Morant (1934) lay down no method for direct measurement of the transverse flattening of the middle part of the facial skeleton, i.e. of the part made up by the malar bones and corpora of the maxillae. In studying a number of crania with differently shaped facial skeletons, it occurred to me that the best expression of the degree of projection of the middle part of the facial skeleton might be obtained by expressing the projective distances of the zygomaxillary and zygotemporal points from the most posterior points on the margins of the pyriform aperture as percentages of the lengths of the chords between the corresponding bilateral points.

It is true that no precise points of general validity for this purpose can be indicated on the "nasolateral" margins. It is also the case that we often find asymmetry here, though hardly more than in other parts of the facial skeleton on which routine measurements are taken. However, the advantages offered by the most posterior points on the margins for subtense measurements appear to me to outweigh the disadvantages. It is a factor of some importance that the two subtense planes in question are almost coincident and approximately horizontal.

After having taken some test measurements with a pair of ordinary co-ordinate callipers, I came to the conclusion that this instrument was unsuitable for my purpose. I therefore decided to undertake a preliminary research with the object of testing the value of the method, disregarding the disturbing factor of asymmetry. For this purpose I designed, and ordered from P. Hermann of Zürich, a special pair of callipers with two parallel arms, both of which might be moved in directions at right angles to the bar to give readings of the distances of the tips from the bar. According to my design the tips were to be slightly blunt, and the bar was to be only 2 mm. thick with blunt edges on its working-face. Owing to the trouble and expense involved in the construction of an instrument of this kind, P. Hermann sent for my approval one of a set of twelve pairs of callipers which he had made some years ago to the order of R. Pöch, some of which were still in his possession (Fig. 1). In most respects the design is similar to my own, but the bar is 3 mm. thick and the edges of the working-face are sharp. Furthermore, the scales of the arms are fixed in such a manner as to necessitate correction of the readings of the subtenses, whereby 4·3 mm. (controlled by the Weights and Measures Standards Office,

Oslo) must be added to the readings. In spite of these drawbacks I decided to keep the instrument, and; though it is not ideal for the purpose, it can be used effectively. I do not know for what kind of measuring it was originally designed.

For measuring a chord and its subtense I first set both arms of the instrument at equal lengths, longer than the subtense, and place the tips in contact with the extremities of the transverse chord. I then fasten the screw, draw back the arms, place the working-face of the bar lightly but firmly against the nasolateral margins, and move the arms until the tips again meet the bilateral points. On removing the instrument the distance between the arms and the subtenses are recorded, the latter after addition of 4·3 mm. If the readings on the two arms are not equal, as

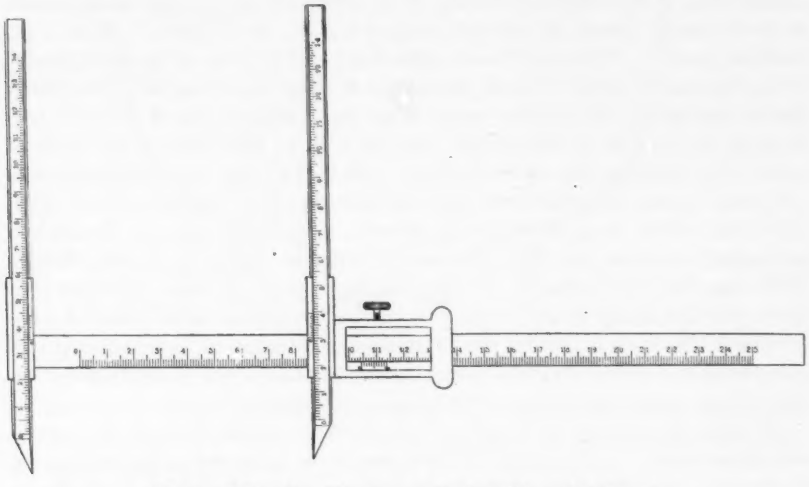

Fig. 1. Callipers used to measure the breadths and subtenses.
(The figures on the scale are centimetres.)

is most frequently the case, the average of the two values is recorded. For my method of measuring, however, I do not regard as suitable skulls with conspicuous asymmetry of the facial skeleton.

With my instrument I also took the measurements which give the "frontal index of facial flatness" of Woo & Morant in such a manner that both arms were of the same, or practically the same, length. This frequently necessitated re-measuring. I do not believe that the length of the subtense obtained by my method differs from that arrived at by using a pair of co-ordinate callipers of the usual form.

My measurements are as follows:

(i) The chord *IOW*, inner biorbital breadth, *fmo-fmo* of Martin (1928).

(ii) *Sub. IOW*, the subtense of the nasion from the chord *IOW*.

(iii) The chord *GB*, bimaxillary breadth, *zm-zm* of Martin.

(iv) *Sub. GB*, the subtense of the nasolateralia (*Nl-Nl*), i.e. the line joining the most posterior points on the margins of the pyriform aperture, from the chord *GB*.

(v) The chord *ZB*, bizygotemporal breadth, between the two *zygotemporalia inferioria* (*ZT* of Woo, 1937).

(vi) *Sub. ZB*, the subtense of *Nl-Nl* from the chord *ZB*.

(vii) *FB*, the bizygomatic breadth, *zy-zy* of Martin.

Before passing on to the indices, I must comment on the fact that the location of the zygomaxillary point (*zm*) gave me a certain amount of trouble. I did not always adhere to Martin's definition of it, which specifies absolutely the lowest point on the zygomatico-maxillary suture. Woo & Morant adopt this definition, adding that "if the inferior extremity of the suture is a short length lying parallel to the horizontal plane, the anterior point on it is the one accepted". What is the position, however, if the length is not quite short, and if it does not lie quite parallel to the horizontal plane, as when the margin is rough and irregular? The fact is that we are dealing here with a region of the facial skeleton which reveals a particularly high degree of variability, due primarily to differences in the mode of origin of the anterior part of the masseter muscle. In many cases the origin does not reach the maxilla, and where this is so location of the point presents no difficulty; but in other cases, and probably for some races in the majority of cases and particularly in males, the origin continues for different lengths on the lower border of the maxilla, and it often shows fairly strong impressions there. This last condition was frequently found in the case of the Norwegian skulls which I have examined. The same is true for the Eskimo and Australian specimens, but it was found much less frequently in the case of the skulls of Lapps and those of some other races. There appear to be racial differences in this respect.

In some cases the lowest point on the suture lies rather far back on a broad and rugged surface. Such a point would appear to be quite useless for the purpose of measuring the projection of the facial skeleton, and in fact it may not always be possible to reach it with the tip of one of the arms of the callipers when the bar is placed on the nasolateral margins. In view of this I was compelled to draw up a new definition of the zygomaxillary point, viz. the lowest point of the zygomatico-maxillary suture which still lies on the anterior surface of the bones. The adoption of this definition may occasionally alter the length of the chord *GB* by a small amount, but this is of little significance in comparison with the change which it will sometimes make in the length of the subtense.

The definition of the zygotemporal point also seems rather uncertain. Woo (1937) defines *ZT* as "the lowest point on the zygomatic suture which is still on the lateral surface of the arch". In some cases, however, it is very difficult to say where the lateral surface of the arch ends. The section of the arch shows considerable variation in form. Rectangular sections which make possible an absolutely precise location of the point are somewhat scarce. More frequently the zygoma has a lateral and a latero-inferior surface which are more or less indistinctly separ-

ated from each other. In such cases I have located the point to the best of my ability on the borderline between the two. In most cases I have found the chord to be 2–3 mm. shorter than the greatest possible breadth between the sutures on the two sides. Although the question is of much less importance here than it is for the zygomaxillary point, a more precise definition is nevertheless desirable.

Should the method which I suggest for measuring facial projections in relation to the nasolateral margins be generally adopted by craniologists, it will be necessary for them to find indisputable definitions to give the exact location of the extremities of the two chords in question.

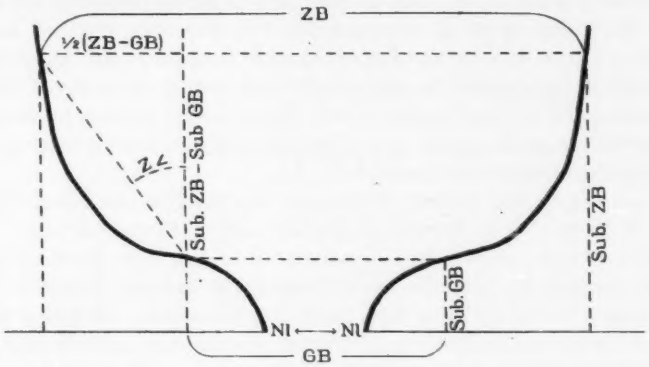

Fig. 2. A horizontal section of the facial skeleton illustrating measurements taken.

The indices used can be divided into two classes, the first (Nos. viii–xi) involving subtenses and the second (Nos. xii–xvi) being ratios of pairs of the transverse breadths. They are:

(viii) *SFi*, 100 *Sub. IOW/IOW* = frontal index of facial flatness,
(ix) *SMi*, 100 *Sub. GB/GB* = maxillary index of facial flatness,
(x) *SZi*, 100 *Sub. ZB/ZB* = zygotemporal subtense index,
(xi) *SSi*, 100 *Sub. GB/Sub. ZB*
(xii) *GOi*, 100 *GB/IOW* = maxillo-orbital breadth index,
(xiii) *ZOi*, 100 *ZB/IOW* = zygomatico-orbital breadth index,
(xiv) *GZi*, 100 *GB/ZB* = maxillo-zygotemporal breadth index,
(xv) *GFi*, 100 *GB/FB* = maxillo-facial breadth index,
(xvi) *ZFi*, 100 *ZB/FB* = zygomatico-facial breadth index.

Finally, I have calculated approximately the angle between the vertical plane through the zygomaxillary and zygotemporal points on either side and the median sagittal plane (see Fig. 2). Assuming that the two planes *zm-Nl-Nl-zm* and *ZT-Nl-Nl-ZT* are horizontal and coplanar, the fraction *(ZB-GB)/2(sub. ZB-sub. GB)* is the tangent of the "zygomatic angle" (*Z∠*).

A metrical description of the curvature of the inferior margin of the malar bone would be of interest, but my attempts to measure it have been unsuccessful. A survey of such a feature would appear to be possible only by using complicated projective methods.

2. The cranial series measured

My main material consists of 100 male and 100 female Norwegian skulls from medieval churchyards in Oslo, and 100 male and 100 female skulls of Lapps obtained from various cemeteries in the county of Finmark, the majority of these being of fairly recent date. Most of the Lapp skulls form part of the material already dealt with by K. E. Schreiner (1931–5), but some of them have been acquired at a later date by the Anthropological Institute, Oslo. All of these are brachycephalic specimens. The Oslo skulls form part of the material which has been described by the same author (1939). I have looked through his records and have omitted the small number of skulls with a cephalic index greater than 79·9, or an upper facial index less than 50·0.

The series of foreign skulls in the possession of our Institute are all too small, and for the most part of too miscellaneous a nature, to provide results of any importance. I have, nevertheless, examined the best of them for the purpose of obtaining comparative data. I measured twenty-five male and twenty-five female Eskimo skulls. The majority of these came from Greenland, but two of the male and six of the female specimens are from the opposite coast of Labrador. I have found no distinct differences between the two local groups. The other series which I measured are:

Ten male and seven female Indian skulls from different parts of America, all of which have common features while none shows artificial deformation;

Eleven male and eleven female Negro skulls from different parts of Africa, all being clearly dolichocephalic;

Thirteen male and nine female native Australian skulls from different parts of the continent;

Twelve male and nine female Maori skulls from a single cave on the North Island of New Zealand.

The Australian and Maori skulls form part of the material dealt with by K. Wagner in his great work (1937). I have only included skulls sufficiently complete to provide all the measurements.

3. Sexual comparisons

Table I gives all the means which I have calculated. We will first examine the question of differences between the sexes. In Tables IIA and B sex ratios expressing the female means as percentages of the male are given for all the characters and groups, and in the lower sections of the same tables the differences between the

TABLE I

Mean measurements

Sex	No.	Series	Absolute measurements							Indices									Angle
			IOW	*Sub. IOW*	*GB*	*Sub. GB*	*ZB*	*Sub. ZB*	*FB*	*SFi*	*SMi*	*SZi*	*SSi*	*GOi*	*ZOi*	*GZi*	*GFi*	*ZFi*	*Z∠*
♂	100	Norwegian	97·71	19·23	95·61	20·21	126·98	48·27	132·84	19·68	21·14	38·01	41·79	97·85	129·95	75·30	72·02	95·60	29°·2
	100	Lapp	98·18	17·53	96·33	17·84	130·64	44·18	135·00	17·91	18·58	33·87	40·41	98·30	133·02	73·75	71·51	96·79	33°·0
	25	Eskimo	99·48	13·62	103·08	11·34	138·68	45·89	142·00	13·69	10·99	33·09	24·70	103·62	139·40	74·33	72·60	97·66	27°·5
	10	Indian	100·30	17·48	103·50	19·59	135·40	49·95	140·20	17·43	18·93	36·89	39·29	103·19	135·00	76·44	73·82	96·59	27°·7
	11	Negro	100·36	17·34	97·91	19·20	127·27	52·43	130·09	16·97	19·61	41·20	36·62	97·56	126·81	76·93	75·26	97·83	24°·4
	13	Australian	104·31	18·71	97·62	20·13	133·54	51·94	136·77	17·94	20·62	38·89	38·76	93·58	128·02	73·10	71·37	97·64	29°·6
	12	Maori	98·83	16·49	100·33	19·98	135·33	49·13	140·67	16·69	19·85	36·25	40·49	101·57	137·18	74·17	71·37	96·21	31°·1
♀	100	Norwegian	94·33	18·32	91·11	18·84	119·74	45·36	124·22	19·37	20·66	37·89	41·52	96·69	126·91	76·14	73·39	96·39	28°·2
	100	Lapp	95·14	16·29	92·24	15·89	122·81	42·67	126·51	17·12	17·23	34·74	37·24	96·91	129·08	75·11	72·91	96·92	29°·8
	25	Eskimo	95·72	13·09	98·44	10·71	128·64	44·21	131·32	13·69	10·88	34·37	24·22	102·84	134·39	76·52	74·97	97·96	24°·4
	7	Indian	94·57	15·97	97·29	18·14	124·29	46·69	129·43	16·99	18·65	37·57	38·85	102·88	131·43	78·28	75·17	96·03	25°·3
	11	Negro	94·27	15·99	91·55	19·24	116·81	47·51	119·82	16·96	21·02	40·67	40·49	97·16	123·90	78·37	76·43	98·25	24°·4
	9	Australian	97·44	16·46	88·00	16·43	122·89	48·80	125·22	16·89	18·67	39·71	33·67	90·38	126·11	71·61	70·28	98·15	28°·3
	9	Maori	95·00	14·18	95·00	17·76	127·33	46·16	131·89	14·92	18·69	36·27	38·47	100·00	133·71	74·70	72·19	96·55	29°·8

TABLE IIA

Sex ratios (female mean/male mean) and mean differences for the absolute measurements and Z∠

(i) Sex ratios

Series	*IOW*	*Sub. IOW*	*GB*	*Sub. GB*	*ZB*	*Sub. ZB*	*FB*	*Z∠*
Norwegian	96·5	95·3	95·3	93·2	94·3	93·9	93·5	96·6
Lapp	96·9	92·9	95·8	89·1	96·7	96·6	93·8	90·3
Eskimo	96·2	96·1	95·5	94·4	92·8	96·3	92·5	88·7
Indian	94·3	91·4	94·0	92·6	91·8	93·5	92·3	91·3
Negro	93·9	92·2	93·5	100·2	91·8	90·6	92·1	100·0
Australian	93·4	88·0	90·2	81·6	92·0	94·0	91·5	95·8
Maori	94·3	91·4	94·0	92·6	91·8	93·5	92·3	91·3

(ii) Sexual differences with standard errors

Series	*IOW*	*Sub. IOW*	*GB*	*Sub. GB*	*ZB*	*Sub. ZB*	*FB*	*Z∠*
Norwegian	3·38 ± 0·52	0·91 ± 0·32	4·50 ± 0·68	1·37 ± 0·40	7·24 ± 0·52	2·91 ± 0·47	8·62 ± 0·56	1°·00 ± 0·66
Lapp	3·04 ± 0·55	1·24 ± 0·30	4·09 ± 0·62	1·95 ± 0·35	7·83 ± 0·59	1·51 ± 0·46	8·49 ± 0·65	3°·2 ± 0·61
Eskimo	3·76 ± 0·90	0·53 ± 0·50	4·64 ± 0·88	0·63 ± 0·98	10·04 ± 1·39	1·68 ± 0·97	10·68 ± 1·42	3°·1 ± 0·97

TABLE IIB

Sex ratios (female mean/male mean) and mean differences for the indices

(i) Sex ratios

Series	*SFi*	*SMi*	*SZi*	*SSi*	*GOi*	*ZOi*	*GZi*	*GFi*	*ZFi*
Norwegian	98·4	97·7	99·7	99·3	98·8	97·7	101·1	101·9	100·8
Lapp	95·5	92·7	102·6	91·9	98·6	97·0	101·9	102·0	100·1
Eskimo	100·0	99·0	103·9	98·1	99·2	96·3	102·9	103·3	100·3
Indian	97·6	98·1	101·8	98·8	99·7	97·4	102·4	101·8	99·4
Negro	100·0	107·2	98·7	110·1	99·6	97·7	101·9	101·5	100·4
Australian	94·1	90·5	102·1	89·5	96·7	98·5	98·0	98·5	100·7
Maori	89·4	94·2	100·1	95·0	98·5	97·5	100·7	101·1	100·3

(ii) Sexual differences with standard errors

Series	*SFi*	*SMi*	*SZi*	*SSi*	*GOi*	*ZOi*	*GZi*	*GFi*	*ZFi*
Norwegian	0·31 ± 0·32	0·48 ± 0·41	0·12 ± 0·41	0·27 ± 0·83	1·16 ± 0·67	3·04 ± 0·62	−0·84 ± 0·49	−1·37 ± 0·51	−0·79 ± 0·23
Lapp	0·79 ± 0·27	1·35 ± 0·37	−0·87 ± 0·37	3·17 ± 0·68	1·39 ± 0·65	3·94 ± 0·66	−1·36 ± 0·42	−1·40 ± 0·46	−0·13 ± 0·18
Eskimo	0·00 ± 0·50	0·11 ± 0·64	−1·28 ± 0·74	0·48 ± 1·34	0·78 ± 1·28	5·01 ± 1·04	−2·19 ± 0·93	−2·37 ± 0·82	−0·30 ± 0·27

means for the two sexes, together with the standard errors of these quantities, are provided for the Norwegian, Lapp and Eskimo series. As will be seen from Table IIA, all the male means of absolute measurements are greater than the corresponding female means, except in the case of *Sub. GB* for the Negro series. Some of the differences for the three longest series are not statistically significant. This is so for *Sub. IOW* in the case of the Norwegian and Eskimo series, although the same difference for Lapps is markedly significant. This is an unexpected conclusion and no great reliance can be placed on it, particularly in view of the fact that K. E. Schreiner has found that in general sexual differences are smaller in Lapps than in Norwegians. The peculiarity noted is undoubtedly due to the mixed composition of the Lapp material. Our Lapps do not form a homogeneous population, being mixed at different places in different degrees with Norwegians and Finns (Quains). The skulls were collected in various localities spread over a wide area. Several of the component local series are small and the sexes are unequally represented in them. I have calculated the sex ratios of this character separately for all the local groups and have found fairly large differences. Some of the group means (Kautokeino, Karasjok and Kistrand) give low sex ratios, due, presumably, to the fact that the male means are too high to be characteristic of pure Lapps.

The last column of Table IIA shows sexual comparisons of the zygomatic angle. In all series except the Negro the female mean is distinctly smaller than the male, although the difference is insignificant in the case of the Norwegian series.

Sexual comparisons of the indices (Table IIB) are of greater interest. In accordance with the results of Woo & Morant, the frontal index of flatness (*SFi*) shows no significant differences, although it appears to have a slight tendency to be lower in the female sex. This is also true for the maxillary index of flatness (*SMi*), whereas the zygotemporal subtense index (*SZi*) shows a slight tendency to be greater in the female sex (cf. the relations of *Z*∠). The means of the *SS* index indicate, in accordance with those of the *SM* index, that the maxillary region tends to be somewhat flatter in females than in males. The values of the different breadth indices show very small sex differences, but nevertheless they show fairly consistent relationships. In all groups the biorbital breadth is a little larger relative to the bimaxillary breadth in females than in males. The three breadth measurements of the middle part of the facial skeleton (*GB*, *ZB* and *FB*) show a slight tendency to be relatively larger anteriorly than posteriorly in female compared with male skulls (cf. *Z*∠). In my material this relation applies particularly to the Eskimos. It may be noted that for their larger series Woo & Morant found a distinctly lower mean than I have for the bimaxillary breadth (*GB*) in female Eskimo skulls. The sex ratio of this measurement in their material is only 92·7, as against 95·2 for the biorbital breadth (*IOW*).

4. RACIAL COMPARISONS

Of the breadth measurements, the internal biorbital (*IOW*) shows the smallest differences between the series and the bizygotemporal (*ZB*) the largest. If the short Negro and Australian series are disregarded, the smallest means of all the breadth measurements are found to be the Norwegian. Woo & Morant give a slightly lower mean for the biorbital breadth, and a considerably lower mean (3·4 mm. less) for the bimaxillary breadth, for their nineteen Norwegian skulls than my values for the medieval Oslo series. The values found by these authors for male Anglo-Saxon and medieval English skulls are slightly higher for the biorbital breadth, and slightly lower for the bimaxillary breadth, than my Norwegian values. With regard to the two breadth measurements in question, there is little difference between the Oslo and Lapp series, particularly in the case of the male skulls. It may be noted that for the 140 male Lapp skulls K. E. Schreiner has given a mean bimaxillary breadth of 95·5 mm., which is 0·8 mm. less than my value for male Lapp skulls and practically the same as my value for male Norwegian skulls. This author has also calculated for 121 female skulls a mean (91·7) which is slightly lower than mine. The difference between our values may be due to some extent to a difference in locating the zygomaxillary point. However that may be, the differences between my female means for Norwegian and Lapp skulls cannot be considered statistically significant in the case of either *IOW* (0·81 ± 0·51) or *GB* (1·13 ± 0·56). The bizygotemporal breadth (*ZB*), however, is significantly greater in the Lapp than in the Oslo skulls. The means of all the breadth measurements of the middle facial skeleton are clearly greater for Eskimos than for Lapps, and the biorbital breadth shows differences of the same sign, though they are much smaller. It is worthy of note that this last breadth is greater, or at all events not smaller, in the remarkably narrow-headed Eskimos than in the broad-headed Lapps. The bimaxillary breadth is probably greater in Eskimos than in any other human race, and among the known peoples of the earth only the American Indians appear to approach them in this respect.

Just as the breadth measurements increase from Norwegian to Lapp and from Lapp to Eskimo type, so, to a still greater degree, do the subtenses to *IOW* and *GB* decrease. Consequently the indices derived from these measurements show marked differences. My means of *Sub. IOW* and of the frontal index of flatness for male Oslo skulls are higher than those found by Woo & Morant for their Norwegian series, and they accord better with the values given by these authors for Swedes. On the other hand, their means of the two measurements for Eskimos are greater than mine.

With their exceedingly low means for the index of maxillary flatness (*SMi*), the Eskimo differs greatly from the other groups examined, including the Indian which otherwise bear some resemblance to the Eskimo. In fact, I know of no

characters more capable of indicating the peculiarity of the Eskimo skull than this index taken together with the chief cranial indices. Nevertheless, the Eskimos are extreme in nearly all characters, and the question remains of the extent to which the index can be considered of value for racial classification in general. I can only contribute a little towards the solution of this problem by comparing my Norwegian and Lapp skulls. The means of the index of maxillary flatness reveal the following differences: ♂ 2·56, ♀ 3·43, with standard errors of 0·40 and 0·38, respectively. The differences are of marked significance.

The zygotemporal subtense (*Sub. ZB*) and its index (*SZi*) show entirely different relations. They are associated less with the "flatness" of the facial skeleton than with the antero-posterior lengths of the calvaria, and facial skeleton. Among the groups which I have examined the Lapp skulls of both sexes have the smallest means for the subtense. The Eskimo values are distinctly higher for the subtense, but owing to its considerable zygotemporal breadth the type has the lower index. The Norwegian skulls have distinctly higher means, both for the absolute measurement and for the index, than Lapps and Eskimos, but their means are exceeded by those for the prognathous Negro and Australian skulls.

The index *SSi*, which gives *Sub. GB* as a percentage of *Sub. ZB*, shows the highest means for the Oslo skulls and, as a matter of course, very low means for the Eskimo skulls.

Among the indices which relate the different breadth measurements to one another, the maxillo-orbital breadth index (*GOi*) would appear to be of value for racial classification, as it appears to differentiate families of races. Judging from the means given by Woo & Morant for the biorbital and bimaxillary breadths the index lies below 100—that is to say, the former is greater than the latter breadth—in all European, southern Asiatic, the Australian and the majority of African populations, while it exceeds 100 in the case of eastern Asiatic, the majority of American, and a few African and Oceanic populations. As regards my material, the Lapps appear to deny their presumed Mongolian origin, since their values for this index do not exceed the Norwegian to a significant extent, while the Eskimos and Indians have means distinctly above 100.

The *ZO* index rises considerably on passing from the Norwegian to the Lapp and then to the Eskimo series. As regards the other indices of breadth, I will merely refer to the low values for the ratio of the bimaxillary to the facial breadth (*GFi*) of Lapps, who are not only distinguished by their weak mandibles, but also by their weak maxillary bones in contrast to those of Eskimos.

Finally, something must be said of the zygomatic angle. As the calculation of this is based on four different measurements, its value may be influenced by as many errors. Moreover, the two subtenses which are used do not lie in exactly the same plane. Even granting these defects, the angle, nevertheless, is fairly expressive as illustrating a feature which is not measured by the indices. I have previously dealt with the sexual differences, and will here confine myself to calling

attention to the difference between the means for Lapps and Eskimos. Both these types are characterized by broad and flat faces with what are called "high cheek bones". The low values for Eskimos may appear to be unreliable, but doubts as to this will disappear on inspection of the skull of an Eskimo from the base. The unusual breadth of the Eskimo face is due partly to the great bimaxillary breadth, and partly to the considerable curvature of the malar bones, but the posterior parts of these bones are long and the zygomatic arches only protrude slightly in a lateral direction. The bimaxillary breadth of the Lapps, on the other hand, is distinctly smaller, the malar bones are rather short in a transverse direction and the arches protrude much more laterally.

5. VARIABILITIES

In Tables IIIA and B will be found the standard deviations, while Tables IVA and B give the coefficients of variation, for the Norwegian, Lapp and Eskimo series. The bimaxillary (*GB*) is the most variable breadth and the bizygotemporal (*ZB*) is the least variable, while the bimaxillary is the most variable subtense—judging by coefficients of variation—and the bizygotemporal is the least variable. In the case of the subtense indices, also, the index of maxillary flatness (*SMi*) varies most, while the zygotemporal index varies least. These relations are undoubtedly due to the great variation in the form of the facial skeleton in the region of the zygomaxillary suture, which affects all the indices involving the bimaxillary breadth. As a test of sexual and racial differences in variability I am restricted to my Oslo and Lapp material. With regard to this question, I am bound to admit that, as long as we cannot count upon absolute accuracy in sexing skulls, very little emphasis can be laid upon any difference found. In my material male variability tends on the whole to be slightly greater than female, both as regards absolute measurements and indices, except in the case of *Sub. IOW*.

Table V gives the average coefficients of variation in the two male and two female series for three breadth measurements (*IOW*, *GB* and *ZB*), for the three corresponding subtenses, for the three corresponding subtense indices (Nos. viii–x), and for the three corresponding breadth indices (Nos. xii–xiv). Averages are also given for all six absolute measurements, for all six indices and, finally, for all twelve characters. At the bottom of the table the corresponding averages are recorded for the male plus the female constants. It will be observed that most of the male averages slightly exceed the female. As regards racial differences, the average for all six absolute measurements is greater for Lapps, but the average for all six indices is greater for Norwegians. All the differences between the averages for the two series are, however, very small.

TABLE IIIA

Standard deviations for the absolute measurements and Z∠

Sex	Series	*IOW*	*Sub. IOW*	*GB*	*Sub. GB*	*ZB*	*Sub. ZB*	*FB*	*Z∠*
♂	Norwegian	3·44 ± 0·24	2·18 ± 0·15	5·49 ± 0·39	3·15 ± 0·22	3·66 ± 0·26	3·41 ± 0·24	4·33 ± 0·31	5°·32 ± 0·38
	Lapp	4·33 ± 0·31	2·15 ± 0·15	4·77 ± 0·34	2·56 ± 0·18	4·46 ± 0·32	3·55 ± 0·25	4·80 ± 0·34	4°·52 ± 0·32
	Eskimo	3·42 ± 0·48	2·00 ± 0·28	2·52 ± 0·36	4·31 ± 0·61	5·79 ± 0·82	4·22 ± 0·60	5·90 ± 0·83	3°·84 ± 0·54
♀	Norwegian	3·96 ± 0·28	2·34 ± 0·17	3·99 ± 0·28	2·53 ± 0·18	3·70 ± 0·26	3·27 ± 0·23	3·61 ± 0·26	3°·89 ± 0·27
	Lapp	3·41 ± 0·24	2·07 ± 0·15	3·91 ± 0·28	2·43 ± 0·17	3·88 ± 0·27	3·00 ± 0·21	4·44 ± 0·32	4°·10 ± 0·29
	Eskimo	2·93 ± 0·41	1·51 ± 0·21	3·62 ± 0·51	1·83 ± 0·26	3·87 ± 0·55	2·47 ± 0·35	3·96 ± 0·56	2°·93 ± 0·41

TABLE IIIB

Standard deviations for the indices

Sex	Series	*SFi*	*SMi*	*SZi*	*SSi*	*GOi*	*ZOi*	*GZi*	*GFi*	*ZFi*
♂	Norwegian	2·21 ± 0·16	2·92 ± 0·21	3·04 ± 0·22	6·49 ± 0·46	4·94 ± 0·35	4·67 ± 0·33	4·07 ± 0·29	4·16 ± 0·29	1·96 ± 0·14
	Lapp	1·91 ± 0·14	2·77 ± 0·20	2·79 ± 0·20	4·82 ± 0·34	4·91 ± 0·35	4·77 ± 0·34	2·89 ± 0·21	3·41 ± 0·24	1·35 ± 0·10
	Eskimo	1·92 ± 0·27	2·69 ± 0·38	3·17 ± 0·45	5·36 ± 0·76	3·90 ± 0·55	4·26 ± 0·60	2·83 ± 0·40	2·47 ± 0·35	1·14 ± 0·16
♀	Norwegian	2·32 ± 0·17	2·82 ± 0·20	2·77 ± 0·20	5·22 ± 0·37	4·47 ± 0·32	4·14 ± 0·29	2·68 ± 0·19	2·87 ± 0·20	1·11 ± 0·08
	Lapp	1·91 ± 0·14	2·51 ± 0·18	2·41 ± 0·17	4·84 ± 0·34	4·23 ± 0·31	4·53 ± 0·32	3·05 ± 0·22	3·07 ± 0·22	1·20 ± 0·09
	Eskimo	1·67 ± 0·23	1·76 ± 0·23	1·95 ± 0·27	4·05 ± 0·57	5·08 ± 0·72	2·97 ± 0·42	3·08 ± 0·44	3·32 ± 0·47	0·74 ± 0·10

TABLE IVA

Coefficients of variation for the absolute measurements

Sex	Series	*IOW*	*Sub. IOW*	*GB*	*Sub. GB*	*ZB*	*Sub. ZB*	*FB*
♂	Norwegian	3·52 ± 0·25	11·34 ± 0·80	5·75 ± 0·41	15·59 ± 1·10	2·88 ± 0·20	7·07 ± 0·50	3·26 ± 0·23
	Lapp	4·41 ± 0·32	12·20 ± 0·86	4·96 ± 0·35	14·00 ± 0·99	3·41 ± 0·24	8·05 ± 0·57	3·55 ± 0·25
	Eskimo	3·44 ± 0·48	14·66 ± 2·07	2·45 ± 0·35	38·00 ± 5·39	4·19 ± 0·59	9·19 ± 1·30	4·15 ± 0·59
♀	Norwegian	4·20 ± 0·31	12·78 ± 0·90	4·39 ± 0·31	13·43 ± 0·95	3·09 ± 0·22	7·20 ± 0·51	2·91 ± 0·20
	Lapp	3·59 ± 0·25	12·68 ± 0·90	4·25 ± 0·30	15·31 ± 1·08	3·16 ± 0·22	7·04 ± 0·50	3·51 ± 0·25
	Eskimo	3·08 ± 0·44	11·52 ± 1·63	3·67 ± 0·52	17·11 ± 2·42	3·01 ± 0·42	5·29 ± 0·75	3·01 ± 0·42

TABLE IVB

Coefficients of variation for the indices

Sex	Groups	*SFi*	*SMi*	*SZi*	*SSi*	*GOi*	*ZOi*	*GZi*	*GFi*	*ZFi*
♂	Norwegian	11·24 ± 0·80	13·84 ± 0·98	7·98 ± 0·56	15·45 ± 1·09	5·04 ± 0·36	3·59 ± 0·25	5·41 ± 0·38	5·78 ± 0·41	2·06 ± 0·14
	Lapp	10·65 ± 0·75	14·93 ± 1·06	8·24 ± 0·58	11·96 ± 0·85	5·10 ± 0·36	3·59 ± 0·25	3·92 ± 0·28	4·77 ± 0·34	1·40 ± 0·10
	Eskimo	14·02 ± 1·98	24·48 ± 3·46	9·57 ± 1·35	21·68 ± 3·07	3·76 ± 0·53	3·06 ± 0·44	3·81 ± 0·54	3·40 ± 0·48	1·16 ± 0·17
♀	Norwegian	11·97 ± 0·85	13·63 ± 0·96	7·30 ± 0·52	12·56 ± 0·88	4·63 ± 0·33	3·27 ± 0·23	4·43 ± 0·31	3·11 ± 0·22	1·15 ± 0·08
	Lapp	10·69 ± 0·76	14·63 ± 1·03	6·95 ± 0·49	13·02 ± 0·92	4·36 ± 0·31	3·51 ± 0·25	4·06 ± 0·29	4·21 ± 0·30	1·26 ± 0·09
	Eskimo	12·24 ± 1·73	16·16 ± 2·28	5·66 ± 0·80	16·74 ± 2·37	4·94 ± 0·70	2·21 ± 0·31	4·02 ± 0·57	4·43 ± 0·63	0·76 ± 0·11

TABLE V

Average coefficients of variation for different sets of characters

Series	3 Breadths	3 Subtenses	3 Subt. indices	3 Br. indices	6 Absolute measurements	6 Indices	All 12 characters
Norwegian ♂	4·05	11·33	11·02	4·68	7·69	7·85	7·77
" ♀	3·89	11·14	10·97	4·11	7·52	7·54	7·53
Lapp ♂	4·92	11·42	11·27	4·20	8·17	7·74	7·95
" ♀	3·67	11·68	10·76	3·98	7·68	7·37	7·52
Norwegian + Lapp ♂	4·49	11·38	11·15	4·44	7·93	7·79	7·86
" ♀	3·78	11·41	10·87	4·05	7·60	7·45	7·53
Norwegian ♂ + ♀	3·97	11·24	11·00	4·40	7·65	7·70	7·65
Lapp ♂ + ♀	4·30	11·55	11·04	4·09	7·93	7·56	7·74

6. Conclusions

The chief result yielded by my study is that measurements of the subtenses from the most posterior points on the margins of the pyriform aperture to certain other bilateral points would be a valuable addition to the routine technique followed in describing series of skulls. Of these measurements the maxillary index of facial flatness appears to be the most useful, but the maxillo-orbital breadth index also gives comparisons of considerable interest. The adoption of these characters in craniological research will necessitate more precise definitions of the zygomaxillary and zygotemporal points than those used hitherto. Furthermore, a new instrument should be designed for measuring projections from the nasolateral margins.

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TRANSPOSITION OF THE VISCERA AND OTHER REVERSALS OF SYMMETRY IN MONOZYGOTIC TWINS

By E. A. COCKAYNE, D.M., F.R.C.P.

The twins, Eileen and Joan C., were first brought to my notice by Dr Reginald Lightwood, who saw them when they were six years old at the Hospital for Sick Children, Great Ormond Street, and found that one had dextrocardia and the other was normal. He remembered that they were very much alike and thought they were monozygotic. With some difficulty I got into touch with them, and Dr James Graham, Assistant County Medical Officer, Essex C.C., kindly examined the parents and the surviving brothers and sisters and found that in all of them the heart was in the normal position and no cardiac abnormality was present. The other children are: Eric, aged 21; Albert, aged 18; Doris, aged 16; Gladys, aged 12; Betty, aged 11; John, aged 9; and Iris, aged 5 years. One boy died in 1919, aged 3½, of lymphatic leukaemia. The twins, born in August 1924, were 13 when examined. The parents are English and are not blood relations. No other case of transposition of the viscera is known to have occurred in the family.

The mother says the twins weighed 3 lb. at birth and that there were two afterbirths, i.e. each had a separate placenta. Clinically the heart is on the right in Eileen and on the left in Joan, and the size and sounds are normal in both. An electrocardiogram of Eileen taken by Dr J. L. Lovibond at the Middlesex Hospital shows inversion of all waves in lead 1, but that of Joan is normal (see pp. 290, 291). An X-ray of the chest and abdomen and a barium swallow showed dextrocardia with the stomach on the right and the liver on the left side in Eileen, and the normal position of the viscera in Joan.

The twins are very much alike in appearance and, though it is possible to distinguish one from the other when together, there is very little doubt that they are monozygotic. In view of the rarity of such a mirror image condition a number of confirmatory observations were carried out. Eileen is right-handed, while Joan is left-handed. Their handwriting is very much alike, but is unformed, and it is difficult to say how far the resemblance is due to teaching. *B*, *R*, and *r* are formed in exactly the same way in both.

Miss Ida Mann examined them and made the following report:

Eileen. Visual acuity 6/6 right and left. Hypermetropia in the right eye and hypermetropic astigmatism in the left. In both eyes the error is slightly higher than in her twin's. Retinal arterial pattern dissimilar in right and left eyes and does not resemble her twin's; practically no mesodermal pigment, well marked lesser circle and no remains of pupillary membrane. The right eye is the master eye.

Joan. Visual acuity 6/6 right and left. Low-grade hypermetropic astigmatism in both eyes. Axes oblique. Retinal arterial pattern dissimilar in right and left eyes and does not resemble her twin's. Iris pattern the same in both eyes and exactly similar to her twin's, practically no mesodermal pigment, well marked lesser circle and no remains of pupillary membrane. The left eye is the master eye.

Miss Mann says that it is usual to find the retinal arterial pattern different in the two eyes of the same person and in corresponding or opposite eyes of monozygotic twins. According to Viggo Eskelund no two persons show an identical iris pattern, but a broad classification into types is possible and these types are determined genetically. Unfortunately the iris pattern in the twins is a very common one.

Dr Phyllis Kerridge tested their hearing in a sound-proof room and both gave very similar graphs, bone conduction being normal and air conduction low normal.

Dr G. M. Morant took the following measurements in millimetres:

	Eileen		Joan	
Stature	1429		1437	
Left cubit	396		396	
Right cubit	397		396	
Maximum head length	180		184	
Maximum head breadth	136		136	
Head height (an uncertain measurement)	128		125	
Cephalic index	75·6		73·9	
Bizygomatic breadth	121		121	
	R.	L.	R.	L.
Maximum length of ear	56	56	55	57
Length of phalanges with fingers strongly flexed at metacarpophalangeal joints:				
2nd digit	92	92	94	93
3rd digit	99	101	103	102
4th digit	93	95	98	98
5th digit	76	76	75	78

The girls are about 2½ in. below the English average stature for their age and social class. The small differences between measurements suggest that they are monozygotic. The small difference between head lengths (leading to one between the cephalic indices) is probably the most significant. Joan has slightly longer digits than Eileen As all the measurements are subject to errors of at least 1 mm. it is safest to conclude that they do not provide any evidence of asymmetry.

He also took photographs and reported as follows: The photographs are of three kinds.

(I) *Profiles.* The outlines of the faces of the two girls are remarkably similar, their upper lips being unusually short. Eileen's chin is rather longer and straighter than Joan's. No difference between the ears (not all visible) was noted. Both have free lobes. They provide no evidence of asymmetry.

(II) *Full-face.* The photographs are not truly full-face, rather more of the

right than of the left being shown in each case. The outlines are very similar, Joan having a rather shorter and more pointed chin than Eileen. The teeth show a slight anomaly, which is the same in both girls. The central upper incisors are crossed, the left overlapping the right.

A palmaris longus muscle is present in both twins on both sides. Hair colour in both twins matches scale 7. The hair whorl was clockwise in both twins.

Miss David took fingerprints and made the following report:

A glance at Waite's tables shows that radial loops are nearly as common as ulnar loops on the forefinger. The remaining fingers are all ulnar loops, which are the commonest of all patterns. There is a distinct similarity between the counts of these ulnar loops and in the actual pattern. In both twins the actual counts approximate to the median value of the distance of ridges in loops. The thumbs are interesting and at first sight appear very similar, but there is a difference as shown in the counts, and the right thumb of Eileen is like the left thumb of Joan, while the left thumb of Eileen is like the right thumb of Joan. The most striking thing is that Eileen has a radial loop on the forefinger of the right hand, and Joan has a radial loop on that of her left hand, and Eileen has an ulnar loop on the forefinger of the left hand and Joan has an ulnar loop on that of the right hand.

	Right hand		Left hand	
	Joan	Eileen	Joan	Eileen
T	23 C 9	21 C 6	5 C 20	11 C 24
1	UL 13	RL 5	RL 5	UL 13
2	UL 13	UL 14	UL 12	UL 13
3	UL 12	UL 12	UL 12	UL 13
4	UL 13	UL 12	UL 13	UL 10

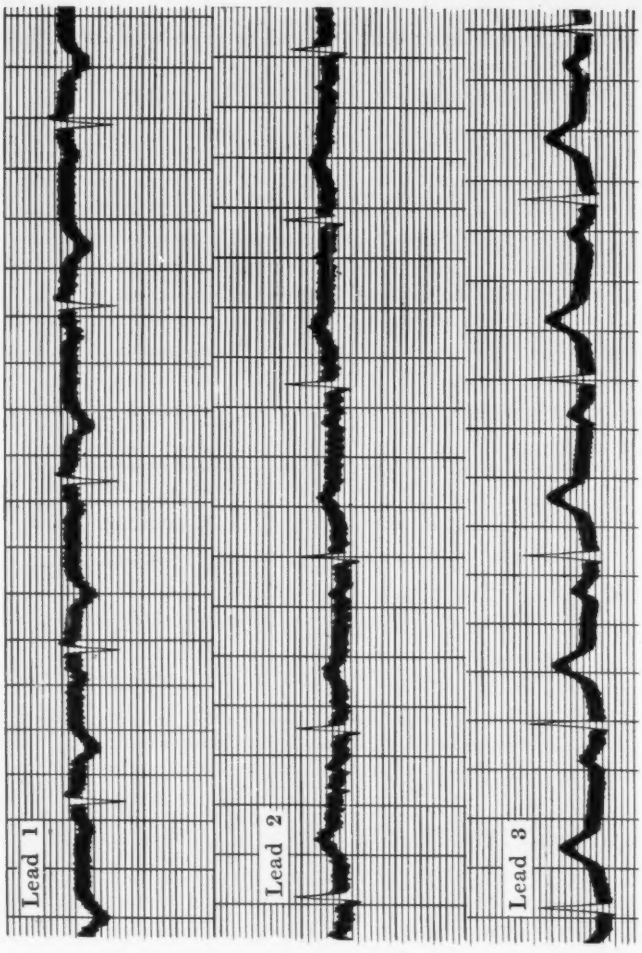
UL = Ulnar loop. RL = radial loop. C = composite.
The numbers are the number of ridges.

The palmar and plantar patterns are similar, but those of Eileen's left side are more like those of Joan's right side than those of her own right side and vice versa.

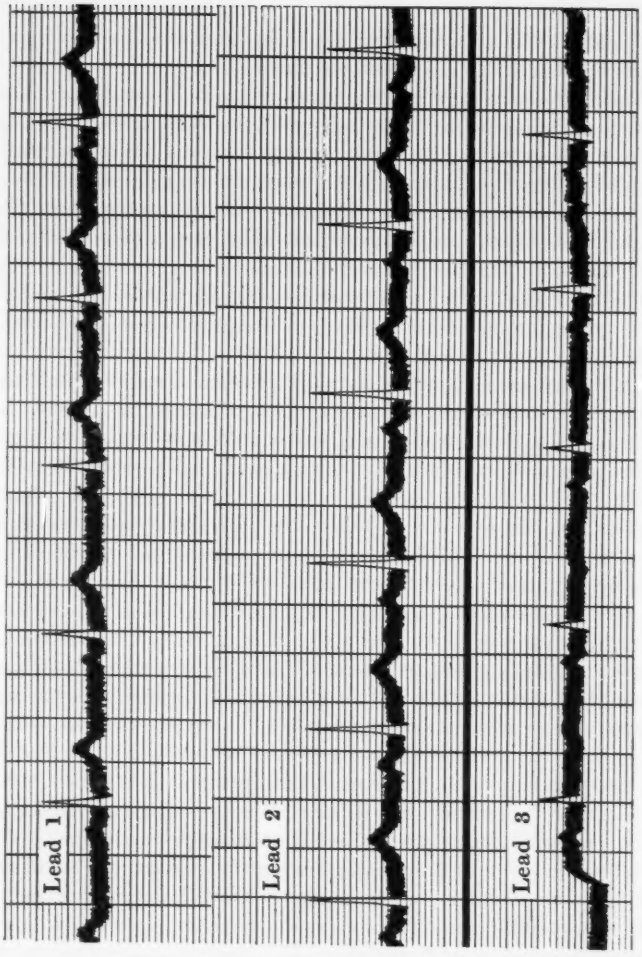
Dr G. L. Taylor finds that the blood groups in both twins are *O M N*. Both are able to taste phenyl-thio-carbamide. The other taste tests were unsatisfactory because they were unable to discriminate clearly between sour and bitter.

At school the twins were in the same class and much alike in mentality. Questioned in different rooms with no opportunity of overhearing one another or comparing notes each said that her favourite colour was blue, one immediately and the other after some hesitation. Neither knew her sister's favourite colour.

I think the resemblance in general appearance and the agreement in so many independent characters, a number of which are known to be determined genetic-



Eileen. Electrocardiograms.



Joan. Electrocardiograms.

ally, proves that Joan and Eileen are monozygotic. They differ from the majority of such twins in the extent to which they are mirror images of one another:

Eileen with complete transposition of the viscera.
Joan with normal viscera.
Eileen right-handed with the right eye the master eye.
Joan left-handed with the left eye the master eye.
Eileen with a radial loop with five ridges on the right forefinger.
Joan with an radial loop with five ridges on the left forefinger.
Eileen with an ulnar loop with thirteen ridges on the left forefinger.
Joan with an ulnar loop with thirteen ridges on the right forefinger.
Eileen with thumb ridges of the right hand closely resembling those of Joan's left hand, and with thumb ridges of the left hand closely resembling those of Joan's right hand.

I can find records of only four similar cases in the literature. Baron (1825) showed one before the Section of Medicine of the Academy of Medicine, Paris, in December 1825. A short report says that the twin with transposed viscera was a girl, who died at the age of 8 days. Kuchenmeister states that the twins were monozygotic and that one had normal viscera and the other complete transposition. Miller (1893) records a case of monozygotic twins, one with situs inversus viscerum and the other with situs solitus, but gives neither their age nor sex. Dubreuil-Chambardel (1927) records male twins, aged 25, identical in general appearance, in weight, height, and in physical characters. Each had varices on the lower limbs of the same type and developing in the same way. One had a harelip on the left, the other on the right, and one had complete transposition of the viscera, while the other was normal in this respect. The visceral condition was confirmed by radiological examination. I have been unable to refer to the original account of the twins recorded by Araki (1934), but Taku Komai, though he gives neither their age nor sex, says they were monozygotic and that one had normal viscera and the other complete transposition. Taku Komai says that this is the fifth case recorded. Possibly he includes that recorded by Tamm and later by Betschler. This refers to female twins, stillborn at the seventh month, one normal, the other oedematous and hydrocephalic, with the heart and aorta on the right side, no lung on the right and a small one on the left. The other viscera were correctly placed, but the liver was small and the kidneys large. There was one placenta with two cords implanted centrally with the umbilical veins anastomosing.

Newman in his *Biology of Twins* shows that the armadillo, *Dasypus novemcinctus*, normally produces monozygotic quadruplets. He makes the following observations. At first there are two embryos and later a secondary embryo is formed to the left of each primary embryo. Asymmetry of the scutes is not uncommon and in the quadruplets all grades of mirror image formation are found. There may be mirror-imaging between individuals of opposite pairs (primary embryos), but this is much less common than imaging between twin

partners derived from one half of the egg (a primary embryo and the secondary embryo to its left). In general mirror-imaging between "opposites" is evidence of a residuum of a primary bilateral symmetry that held sway in the blastocyst before polyembryonic fission began. When the primary outgrowths are formed, they are the product of the antimeric halves of the first embryo and should therefore show mirror image relations. But a partial physiological isolation of the two halves permits a certain reorganization or regulation of new symmetry relations, which tends more or less completely to destroy the original symmetry, yet often leaving a trace of the latter. Finally when each secondary outgrowth organizes its own bilateral symmetry, it tends to lose, partially at least, the earlier symmetry relations, and to establish its own mirror-imaging of right and left halves. In some cases traces of all three symmetry systems appear in a single set of foetuses, but it is common to find only two systems interacting. Newman says that no case of visceral reversal was found in spite of a careful search through a large number of foetuses. This, he thinks, is due to the fact that twinning is initiated and carried out in the ectoderm, and that the endoderm becomes involved only passively and considerably later.

In the case of human monozygotic twins there may be almost complete mirror image arrangement of palmar patterns so that the left hand of x corresponds with the right hand of y and vice versa. This corresponds with the rather rare mirror-imaging of "opposites" seen in armadillo quadruplets and in Newman's opinion goes far to prove that polyembryony actually occurs in man. A commoner manifestation in man is reversal of symmetry in the pattern of an index finger of one twin so that it mirrors the condition in the corresponding index finger of the other twin. This reversal of pattern occurs in both hands of Eileen and Joan.

In addition to the ridges on the skin other epidermal structures may show a mirror image arrangement, such as naevi and angiomata, and Siemens (1924) records an accessory nipple below the normal one, which was on the right side in one and on the left side in the other twin.

Newman believes that monozygotic human twins become physiologically isolated at a considerably earlier period than do armadillo quadruplets and founds his belief on the fact that there is so little mirror-imaging in the former and so much in the latter. He says that as a general rule the earlier the separation the more complete is the reorganization of the symmetry relations in the separate individuals and the less residuum of the original common symmetry.

In double monsters of monozygotic origin there is, according to Dubreuil-Chambardel (1927), frequently a hare-lip, which is axial in both or lateral in both, and as a rule it is of the same degree, and the hare-lip of one component is always a mirror image of the hare-lip of the other. He points out as many others have done how often one component of a double monster has complete or partial transposition of the viscera, the arrangement being that the apices of the two hearts point away from the place where the twins are attached to one another.

There can be no doubt that neither the hare-lip nor the transposition of viscera in double monsters is of genetic origin. Hare-lip in man is sometimes determined genetically, but the incidence in double monsters is far too high for this to be true in their case. Newman says that in the case of double monsters fission must begin much later than usual and is never completed. While in the armadillo polyembryony is the rule and fission is standardized, in man it is the exception and there is considerable variability in the period at which fission takes place. Transposition of the viscera in man is recessive to the normal arrangement, and elsewhere I have suggested (1938) two ways in which it could affect only one member of a pair of monozygotic twins. Apart from the improbability of so unusual an occurrence happening in at least five recorded cases, neither explanation accounts for the occurrence of other mirror image arrangements, and any genetic explanation must be abandoned. If Newman's views about mirror image formation are accepted, the very rare cases of human monozygotic twins with a mirror image arrangement of the viscera in addition to that of ectodermal structures are due to an unusually late fission. The fact that they are so rare seems to show that unless fission is very late each twin reorganizes its own symmetry and, if genetically normal, the viscera are normal in both, but if the zygote is a homozygous recessive for transposition of the viscera, the viscera are transposed in both. Apparently few twins in which late fission occurs succeed in becoming completely separate, for double monsters with their own mirror image symmetry, i.e. with the viscera transposed in one and normal in the other, are far commoner than separate twins with this arrangement. The separate twins, which do show this arrangement have in fact just escaped being double monsters.

My thanks are due to Prof. R. A. Fisher for kindly putting the resources of the Galton Laboratory at my service and to Dr Julia Bell for her help and advice, and to Miss David, Dr Taylor and Dr Morant. I wish also to express my gratitude to Miss Ida Mann and to Dr Lovibond. Dr Graham had intended to write a report on the twins, and I should like to thank him for allowing me to do so instead and for examining the other members of the family.

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THE HUMAN REMAINS OF THE IRON AGE AND OTHER PERIODS FROM MAIDEN CASTLE, DORSET

By C. N. GOODMAN and G. M. MORANT

1. Introduction

Excavations at Maiden Castle, the largest fortified site of prehistoric date in England, were conducted under the direction of Dr R. E. M. Wheeler during four consecutive seasons from 1934 to 1937. The discoveries made were of unusual archaeological interest, and they include a number of skeletal remains which were preserved with unusual care. The bones available represent 104 individuals ranging in age from a foetal to a senile stage of development. This paper is a report on the eighty-three skeletons which are sufficiently well preserved to be of some anthropological interest, and the periods, sexes and age groups to which they are assigned are given in Table I. An unpublished report on the excavations by Dr Wheeler contains particulars of each of the 104 individuals, and a section by the present writers in the same work gives a summary of the general conclusions reached below and fuller descriptions of the mutilated specimens.

TABLE I

The periods, sexes and ages of individuals from Maiden Castle whose skeletons were measured

Period or group	Adult ♂	Adult ♀	Adolescent ♂	Adolescent ♀	Infant	Totals
Neolithic	3	1	—	—	—	4
Iron Age *A*	1	—	—	—	4	5
Iron Age *B*	2	7	1	—	3	13
Iron Age *C*	8	8*	—	1	—	17
Belgic	—	1	—	—	1	2
Belgic War Cemetery	20	9	—	1	1	31
Romano-British	3	3	—	—	—	6
Iron Age or Romano-British	3	1	—	—	—	4
Saxon	1	—	—	—	—	1
Totals	41	30	1	2	9	83

* Including one individual (T 28) classed as Iron Age *C* or Romano-British.

The metrical study of the skeletons made hitherto has been confined to the crania and mandibles, on which all the customary measurements have been taken, and to the clavicles and long bones of the arms and legs, described by

lengths only. Individual measurements and remarks for the crania and long bones are given in the appended Tables VII and VIII; remarks on the mandibles are given with those on the crania, but the individual measurements of these bones are not provided. Contours of the crania have been drawn, but they are not used here, as additional English material of the same periods would be needed to furnish sufficiently reliable type figures. The writers hope to undertake a more detailed examination of the long bones and an examination of the other bones of the skeleton.

TABLE II

Maximum lengths of the shafts of long bones of infants of Iron Age date and of modern foetuses

No. ...		GM I	P III	Q II	R I	P 1	R II	T 19	O	P 35	—	—
Year excavated ...		1936	1937	1937	1937	1937	1937	1937	1936	1937	—	—
Period* ...		*A*	*A*	*A*	*A*	*B*	*B*	*B*	*C*†	*C*‡	Modern§	Modern‖
Femora	R.	—	75·1	—	78·9	—	—	—	—	84·9	65·1	58·0
	L.	—	75·1	—	—	85·2	76·9	72·4	81·5	—	65·0	58·0
Tibiae	R.	—	64·0	—	68·2	—	67·0	68·2	—	—	56·1	51·0
	L.	—	64·2	—	68·8	—	67·1	68·8	—	72·1	56·5	51·1
Fibulae	R.	—	61·0	—	—	—	—	61·2	68·0	—	—	48·2
	L.	—	61·2	—	—	—	—	61·2	—	—	53·0	48·2
Humeri	R.	70·5	64·1	—	66·0	74·0	67·0	66·3	71·4	73·0	58·1	53·1
	L.	70·7	64·8	75·5	67·8	—	—	—	71·4	73·0	58·2	53·0
Radii	R.	—	49·0	—	54·4	58·2	52·7	51·8	55·5	—	45·0	44·0
	L.	—	49·1	—	54·4	58·0	—	51·2	55·7	57·9	45·0	44·0
Ulnae	R.	64·4	58·2	—	—	66·2	61·1	—	—	—	53·0	49·0
	L.	—	58·0	—	61·4	—	—	58·8	64·5	—	53·0	49·0
Clavicles	R.	—	46·0	—	—	48·3	—	42·1	49·1	46·4	39·2	34·5
	L.	47·0	46·0	—	—	48·4	—	42·6	—	—	38·8	34·0

* *A*, *B* and *C* refer to the chronological divisions of the Iron Age. † Belgic.
‡ Belgic War Cemetery. § Full term. ‖ 7 months foetus.

Nearly all the skeletons are incomplete, but in the majority of cases the sexes of the adult and adolescent individuals can be judged, principally from the pelvis, with a reasonable assurance of accuracy. It was suspected at first that nine skeletons, which are remarkably well preserved in view of their tender age, were foetal. The maximum lengths of the shafts of the long bones for these—the measurements having been taken in any direction by using the flat arms of a pair of small calipers—are given in Table II. The last two columns on the right of the table give the same lengths for a child at birth and a seven months foetus. These two specimens are preserved in the museum of the Department of Anatomy, University College, London, and we are indebted to Dr Matthew Young for giving us access to them. All nine of the skeletons from Maiden Castle are appreciably larger than the two of modern date, and hence it must be

supposed that the former are of young infants, probably all from three to six months old. No other measurements of them were taken.

The remaining seventy-four skeletons are available for racial comparisons, and the majority of these are of Iron Age date. Thirty represent the population of the site at the very end of the period, as they came from a cemetery used for the Belgic defenders of the Castle who were slain by Roman invaders in the year A.D. 43. Nearly half of the skulls in this series bear witness to the event in the form of sword-cuts and other mutilations. The Belgic War Cemetery series is just long enough to make statistical comparisons between it and other groups worth while, but all the other groups distinguished in Table I are clearly too small to stand alone. In some comparisons below a composite series made up by pooling all the other individuals of Iron Age or Romano-British date is considered, either apart from the Belgic War Cemetery series or combined with it. This procedure is of provisional value only, of course. There may have been changes in the racial composition of the population of Maiden Castle during the Iron Age and again in Roman times, and the evidence available is quite insufficient to show whether this was so or not.

It should be appreciated, too, that it is not unlikely that a small community, which may have been peculiar owing to relative inbreeding, may have persisted there for several generations, and the characteristics of such a local group may mislead if it is taken to typify the large racial population of which it formed a special part. The new material from Maiden Castle provides a very welcome addition to our meagre knowledge of the physical characters of the inhabitants of England in Iron Age times. It may be hoped that it will form a nucleus to which other specimens of the period—both those already housed in museums and others as yet undiscovered—may be added, until the evidence is abundant enough to satisfy the most exacting anthropologist. The object of this paper is to provide a descriptive record of the skeletons available with such a hope in view, and any results regarding racial relationships made in it are intended to be of a tentative nature only.

2. NON-METRICAL FEATURES OF THE MAIDEN CASTLE SERIES

The remarks on the Iron Age and Romano-British series given in this section relate almost entirely to the skulls. Comments on a few of the long bones which exhibit gross abnormalities are given in Table VIII, and the remaining parts of the skeletons have not been examined. The Neolithic skeleton of a young man (Q 1) was extensively mutilated, and otherwise it is not remarkable. The skull-cap of the same period is also that of a young man. The Saxon skeleton (Q) is male but the age at death cannot be estimated as the skull is missing. The following remarks refer to the remaining specimens, which are all of Iron Age or Romano-British date.

Estimates of the ages at death of the adults were obtained by noting the conditions of the coronal, sagittal and lambdoid sutures, and the ectocranial aspects of these give the following frequencies:

		All sutures open	Sutures beginning to close or partly closed	All sutures closed	Totals
♂	Belgic War Cemetery series	5 (28%)	12 (67%)	1 (6%)	18
	Others	3 (21%)	9 (64%)	2 (14%)	14
	Total series	8 (25%)	21 (66%)	3 (9%)	32
♀	Belgic War Cemetery series	3 (43%)	4 (57%)	—	7
	Others	7 (37%)	10 (53%)	2 (11%)	19
	Total series	10 (38%)	14 (54%)	2 (8%)	26

It is probable that all the people buried in the Belgic War Cemetery were massacred. The age constitutions of the short adult series are very similar, however, to those of the series made up by other people interred at Maiden Castle, who probably died from natural causes. The massacre must have been carried out without regard to age, though there appears to have been some sex discrimination, since twice as many men as women were excavated from the cemetery (see Table I). The percentage frequencies for the total series show the usual sex differences, due to the fact that the sutures tend to close at an earlier age in males than in females. Comparison with values obtained in the same way for other series (see Risdon, 1939, p. 107) shows that the Iron Age men died at a rather younger age, on the average, than those interred in English cemeteries of a later date, but that the women are not distinguished in the same way. In considering the frequencies of anomalies, the total series from Maiden Castle is referred to below.

The sagittal suture was normally the first to close, followed by the coronal and then the lambdoid. Only one specimen (P 23, male) is definitely anomalous with regard to the condition of the principal calvarial sutures. Its sagittal suture is obliterated, the coronal is beginning to close and the lambdoid is open. This specimen shows no apparent sign of distortion and its cephalic index (77·4) is above the average, so it may be assumed that the sagittal suture was not obliterated before maturity was reached. Of thirty-three male frontal bones only one (P ii, an isolated bone for which no measurements are given) is metopic, and there is only one female metopic specimen out of a total of twenty-seven frontal bones. Among Western European crania metopic specimens are usually found with a frequency of about 10%, but the Iron Age series are so short that no significance can be attached to the fact that the frequencies for them are exceptionally low. There are no examples of interparietal bones, or of an *os*

épactal, among the thirty-two male and twenty-six female occipital bones. A much rarer anomaly is exhibited by a male cranium (P 7A). This is a trace of a suture (33 mm. long) extending across the posterior part of the right parietal bone: its position can be seen from Plate II B. One male skull (P 27) shows the occipito-mastoid suture obliterated prematurely on both sides. There is one case of fronto-temporal articulation on both sides among the males, and one of the pterion in *K* on both sides among the females. One female specimen has both malar bones divided by horizontal sutures. A male skull (O 3, Plate III B) has the basi-occipital broken exposing the extension into it of the sphenoidal air sinuses, the condition exhibited by the Swanscombe skull. There is one large cavity and the extremity of a small one to the right of it.

Wounds that had healed completely during life were noted on the cranial vaults of three of the male and two of the female skulls. None of these was severe and a female specimen (T 22) has a depression on the right temple which was more serious than the other injuries. An adolescent female skull (P 20, Plate II A) has the right malar bone deformed, probably as the result of a wound. By far the most serious injuries of traumatic origin are shown by the long bones of one of the males (T 5). His right elbow was shattered, involving complete separation of the proximal extremity of the ulna, his left radius was fractured near the wrist, and his fractured left fibula became fused to the tibia. Healed fractures of the long bones were only found in the case of two other skeletons, viz. P 23 (male, right ulna) and P 36 (female, left fibula). Pathological conditions of bones not of traumatic or dental origin were only noted in the case of one male skull (T 9) which has a swelling on the right maxilla below the orbit which was probably caused by a tumour, of a skeleton of the same sex (P 25) which has osseous deposits on the muscular ridges of the left femur and fibula, and of a female (T 12) which has the left humerus markedly deformed, probably as the result of an inflammatory condition of the surrounding soft tissues. We are indebted to Dr A. M. El Batrawi for help in interpreting the conditions of these and other specimens in the series.

The teeth of the individuals interred at Maiden Castle are by no means well preserved. The following frequencies are found for the total "Iron Age" series:

	Upper jaw		Lower jaw	
	♂	♀	♂	♀
No teeth lost before death	12	8	22	13
One or more teeth lost before death	16	15	10	12

In the case of excavated series of skulls which are not of recent date, it is customary to find for either sex that more than half the specimens had lost no

teeth from the upper jaw before death. Abscess cavities were found in several of the jaws. A female specimen (T 21, Plate III A) has two symmetrically disposed cyst cavities in the buccal surfaces of the maxillae at the roots of the second premolars. There are examples of crowded front teeth. A male mandible (P 23) has a milk molar retained in the position of the left second premolar, which had not appeared, and a female bone (P 26) appears to have had only three incisors erupted. The upper jaw of a female skull (T 28, Plate III D) shows the canines missing and the first premolars rotated. A female mandible (P 14, Plate III E) has the left third molar impacted but almost fully erupted: there are also distinct swellings on the inner alveolar margin extending from the first premolar to the first molar on the left side, and from the second premolar to the first molar on the right. Several of the *Sinanthropus* mandibles exhibit this condition (*torus mandibularis*), and it occurs more frequently, and to a more marked extent, in some modern races of man (particularly in Eskimos), but is seldom found in European series. Dental anomalies appear to be exceptionally frequent in the short series of skulls from Maiden Castle.

In general appearance the specimens show considerable individual differences, but nearly all might be considered quite unexceptional if found in any British collection of later date than the Bronze Age. There are few examples, however, of the markedly retreating frontal bones which characterize the seventeenth-century London skulls, particularly those from Farringdon Street. The variation exhibited appears to be no greater than that expected for a community of intermarrying people, except for the fact that one female skull from the Belgic War Cemetery (P 36, Plate II C, D) stands apart from the others on account of its aberrant form. The facial skeleton has an unusual premaxillary height, though it is not prognathous; the nasal index is high but not extreme, and the nasal bridge is broad and depressed. The cephalic index of this specimen (87·0) is easily the highest for the series, and the individual was decidedly short (1451 mm. = 4 ft. 9¼ in.), though taller than two other women interred in the Belgic War Cemetery. It is possible that the skeleton P 36 is that of an alien in Western Europe, but it appears to be more probable that its peculiarities are of individual rather than racial significance. Its measurements were included in computing averages.

3. Metrical comparisons of the Iron Age and Romano-British crania

Excluding the two Neolithic specimens, there are 30 adult male and 26 adult female crania sufficiently complete to give measurements, though most of these are defective to some extent. Individual readings are in the appended Table VII. In spite of the small numbers, it appeared worth while computing separate means of the more important characters for (*a*) the Belgic War Cemetery series, and

(*b*) all the other specimens of Iron Age or Roman date. The latter is a miscellaneous collection made up by individuals belonging to the following groups:

	Male	Female		Male	Female
Iron Age *A*	1	—	Romano-British	3	3
„ *B*	1	7	Iron Age or Romano-British	2	1
„ *C*	6	8			

The pooling of this material, and treatment of it as if all the specimens represented a single racially homogeneous population, is obviously a *pis aller*. The means found for the composite group (Table III) are actually very close to those found for the Belgic War Cemetery series, which is almost certainly made up by individuals who belonged to a single community. On the supposition that the variabilities of the groups were of the usual order, all the differences between the two sets of means are quite insignificant. Both types have surprisingly large basio-bregmatic heights, and hence unusual indices (100 H'/L and 100 B/H') involving this diameter. Otherwise, they appear to have no features which can be considered at all peculiar in English series of post-Bronze Age date.

This comparison provides some justification for combining the two subgroups, to give a single series which may be supposed to represent the population of Maiden Castle from Iron Age *A* to Roman times, though it is clear that the evidence available is quite inadequate to demonstrate that the racial constitution of the population remained stable throughout the period. The following distributions are found for the total sample.

Cephalic index (central values)	67	69	71	73	75	77	79	81	83	85	87	Totals	
♂	1	1	2	5	6	7	—	1	—	—	—	23	
♀	—	—	2	4	9	4	—	2	—	—	1	22	
Height-length index (central values)	68·5	69·5	70·5	71·5	72·5	73·5	74·5	75·5	76·5	77·5	78·5	79·5	Totals
♂	—	3	1	4	3·5	3·5	2	2	1	—	—	1	21
♀	2	1	1	1	2	4	6	1	2·5	0·5	1	—	22

The highest cephalic index (87·0) is for a female specimen (P 36), in the Belgic War Cemetery series, which is also remarkable on account of the fact that its facial skeleton is of an unusual form. The highest height-length index is for a

TABLE III

Mean measurements of series of Iron Age and Romano-British crania from Maiden Castle

	Male			Female		
	Belgic War Cemetery	Others	Total	Belgic War Cemetery	Others	Total
L	187·7 (14)	189·9 (9)	188·6 (23)	179·0 (7)	180·5 (19)	180·1 (26)
B	141·4 (14)	140·0 (10)	140·8 (24)	135·7 (6)	135·8 (16)	135·8 (22)
H′	137·1 (14)	136·9 (8)	137·1 (22)	134·4 (7)	131·7 (15)	132·6 (22)
B′	97·2 (14)	96·9 (12)	97·1 (26)	92·5 (7)	94·1 (19)	93·7 (26)
S	384·5 (11)	381·9 (7)	383·5 (18)	371·3 (6)	365·6 (10)	367·7 (16)
U	528·6 (12)	530·9 (8)	529·5 (20)	504·0 (6)	508·1 (15)	507·0 (21)
fml	37·8 (13)	37·3 (8)	37·6 (21)	35·9 (5)	35·3 (11)	35·5 (16)
fmb	31·6 (12)	31·2 (8)	31·5 (20)	28·2 (5)	28·9 (10)	28·6 (15)
LB	102·5 (13)	102·2 (6)	102·4 (19)	96·7 (7)	99·9 (12)	98·7 (19)
G′H	73·3 (8)	71·3 (6)	72·4 (14)	68·1 (6)	67·2 (9)	67·6 (15)
NH, L	52·4 (9)	51·2 (7)	51·9 (16)	46·7 (6)	49·1 (12)	48·3 (18)
NB	25·0 (8)	25·1 (8)	25·0 (16)	22·4 (5)	24·3 (8)	23·5 (13)
O_1L	43·6 (9)	41·2 (6)	42·6 (15)	41·3 (5)	41·3 (9)	41·3 (14)
O_2L	33·2 (9)	32·3 (6)	32·9 (15)	32·5 (5)	31·9 (9)	32·1 (14)
100 *B/L*	75·4 (14)	73·5 (9)	74·6 (23)	76·6 (6)	75·5 (16)	75·8 (22)
100 *H′/L*	73·4 (13)	72·4 (8)	73·0 (21)	75·2 (7)	72·9 (15)	73·6 (22)
100 *B/H′*	103·3 (13)	102·7 (8)	103·1 (21)	101·6 (6)	102·5 (13)	102·2 (19)
100 *fmb/fml*	83·5 (12)	83·7 (8)	83·6 (20)	79·2 (5)	82·9 (9)	81·5 (14)
100 *NB/NH, L*	48·6 (7)	50·2 (6)	49·4 (13)	48·1 (5)	50·0 (8)	49·3 (13)
100 O_2/O_1, *L*	76·3 (9)	78·5 (6)	77·2 (15)	78·8 (5)	77·2 (9)	77·8 (14)
P∠	86°·7 (8)	86°·0 (4)	86°·5 (12)	89°·0 (3)	83°·9 (9)	85°·2 (12)

Total series

	Male	Female
$\beta Q'$	316·7 (20)	305·3 (23)
S_1'	114·2 (24)	110·2 (22)
S_2'	116·5 (22)	112·2 (21)
S_3'	99·9 (22)	97·0 (17)
S_1	132·0 (22)	126·6 (22)
S_2	129·9 (21)	125·3 (21)
S_3	122·3 (22)	116·0 (17)
*C**	1514·7 (19)	1380·7 (19)

	Male	Female
B″	122·4 (17)	114·7 (16)
Biast. B	114·3 (19)	107·2 (14)
GL	93·4 (14)	90·3 (11)
GB	95·2 (18)	89·4 (14)
J	133·6 (11)	123·2 (11)
G_1'	45·6 (19)	44·0 (12)
G_2	40·7 (16)	39·0 (11)
SS	4·7 (16)	4·3 (14)

	Male	Female
SC	10·1 (16)	9·3 (15)
Oc.I.	58·7 (22)	60·5 (17)
100 *G′H/GB*	76·3 (13)	73·5 (11)
100 G_2/G_1'	88·7 (12)	87·4 (7)
100 *SS/SC*	47·6 (16)	47·6 (14)
N∠	61°·0 (14)	63°·2 (11)
A∠	76°·3 (14)	75°·2 (11)
B∠	42°·6 (14)	41°·6 (11)

* Reconstructed from *L*, *B* and *H′*.

male cranium (P 7A) in the same series. In spite of these outlying cases, the distributions for the two characters in question, and those for the other characters, provide no clear evidence of racial heterogeneity, though it cannot be inferred from them that the total population considered was racially homogeneous.

If the supposition that all the skulls of Iron Age and Romano-British date from Maiden Castle represent a single population be accepted, the series available is still not long enough to give reliable racial comparisons of a statistical kind. A short series is liable to indicate an absence of differentiation from other series when one of an adequate length from the same population will provide evidence of distinction, and it will also be liable to give a misleading estimate of divergence from other types when distinction is clearly indicated. Large enough samples must be demanded, in particular, when making comparisons between closely related populations, such as the group made up by all the prevailing populations of England from the end of the Bronze Age to modern times. In spite of its restricted size, coefficients of racial likeness were computed for male means between the total "Iron Age" series from Maiden Castle and four others, viz.:

(*a*) The Anglo-Saxon made up principally by skulls preserved in London Museums (Morant, 1926).

(*b*) The so-called British Iron Age (Morant, 1926) made up by skulls from the south of Scotland and various parts of England. Several of the specimens are known to be of Romano-Britons, and some of the others are not dated satisfactorily, so the series is of little value.

(*c*) The seventeenth-century series from a plague in Whitechapel (Macdonell, 1904), the revised means given by Hooke (1926) being used.

(*d*) The seventeenth-century series from the cemetery in Farringdon Street (Hooke, 1926).

The following coefficients of racial likeness are found for the total "Iron Age" series from Maiden Castle and these four, the numbers in brackets following the crude values being the numbers of characters on which they are based, and the numbers in brackets following the names of the series being the average numbers of skulls on which the means used are based ($\bar{n}$'s)*:

	Crude C.R.L.	Reduced C.R.L.
Maiden Castle (17·7) and Anglo-Saxon (34·1)	$-0{\cdot}28 \pm 0{\cdot}18$ (29)	—
" (18·5) and British Iron Age (55·4)	$2{\cdot}98 \pm 0{\cdot}21$ (20)	$10{\cdot}73 \pm 0{\cdot}77$
" (17·7) and Whitechapel (90·7)	$2{\cdot}79 \pm 0{\cdot}17$ (30)	$9{\cdot}43 \pm 0{\cdot}59$
" (17·8) and Farringdon Street (96·7)	$4{\cdot}29 \pm 0{\cdot}17$ (30)	$14{\cdot}27 \pm 0{\cdot}58$

As far as can be told from the scanty evidence, the Maiden Castle and Anglo-Saxon samples might represent different sections of the same population, while the former series is clearly differentiated from the other three. Previous comparisons have shown that the British Iron Age and the two seventeenth-century London types are very similar, while the Anglo-Saxon stands apart from that cluster. These relationships suggested that the Londoners were descended primarily from the pre-Saxon rather than from the Anglo-Saxon population, but the new evidence does not support this view.

* The standard deviations of the Farringdon Street series were used in computing the coefficients.

For characters considered singly, there are very few significant differences between the means for the short Maiden Castle series and those for the three later series. All the differences from the Anglo-Saxon values are quite insignificant, the highest of the 29 α's being 3·8. The only α's greater than 10 are for the nasal angle ($N\angle$, 15·0) and basio-bregmatic height (H', 11·8) in comparisons with the British Iron Age means, for H' (39·7), $100\,H'/L$ (32·7) and $100\,B/H'$ (28·7) in comparisons with the Farringdon Street means, and for H' (18·9), $100\,H'/L$ (15·3), $N\angle$ (15·2) and $100\,B/H'$ (10·5) in comparisons with the Whitechapel means. The Maiden Castle skulls are markedly orthognathous judging from the nasal angle, but this measurement is only available for 14 male skulls and little stress can be laid on the peculiarity. Otherwise the type is only distinguished by a calvarial height which is large both absolutely and also relative to the length and breadth of the brain-box. It is known that the Anglo-Saxon is distinguished in precisely the same way from the other types considered.

Table IV gives male means of the three principal diameters and the three indices derived from them for the series referred to above and the following:

(*e*) Three series of Romano-British skulls described by Buxton (1935), the six measurements in question being the only ones available for these.

(*f*) An Anglo-Saxon series from Burwell, Cambridgeshire (Brash *et al.* 1935).

(*g*) A third seventeenth-century London series from a burial-pit at Moorfields (Macdonell, 1906).

(*h*) A modern series of Lowland Scottish skulls measured by Turner (1903) and compiled in the way described by Hooke (1926, pp. 22 and 38).

(*i*) A modern series from Glasgow (Young, 1931).

All the series included in the table are believed to represent populations which prevailed in England and the south of Scotland at certain periods from the beginning of the Iron Age to modern times, and the minority populations for which there is craniological evidence are omitted. The length, breadth and cephalic index are seen to be remarkably constant throughout, while the basio-bregmatic height and the two indices involving this diameter show larger differences. The greatest heights, highest height-length and lowest breadth-height indices are for the two series from Maiden Castle, and they thus show a remarkable concordance in spite of their very restricted sizes. In these respects one of the Romano-British series (the Brigantes) and the two Anglo-Saxon come next, and the others follow, with one other Romano-British (the Dobuni) and two of the seventeenth-century London series at the lower end of the scale. Little significance can be attached to this order, however, as some of the series are short. There are no statistically significant differences, for example, between the means for the total series from Maiden Castle and those for the Romano-British Belgae. The populations represented in Table IV are all remarkably similar in type, and they must have been closely inter-related. Larger series

TABLE IV

Mean measurements of British series of male skulls

Iron Age and Romano-British series	Maiden Castle			"British Iron Age"	Romano-British		
	Belgic War Cemetery	Others	Total		Belgae	Dobuni	Brigantes
L	187·7 (14)	189·9 (9)	188·6 (23)	187·4 (61)	189·6 (40)	190·8 (85)	189·9 (57)
B	141·4 (14)	140·0 (10)	140·8 (24)	141·4 (102)	141·0 (41)	144·2 (77)	141·7 (57)
H′	137·1 (14)	136·9 (8)	137·1 (22)	132·9 (77)	134·2 (33)	132·5 (45)	136·8 (38)
100 *B*/*L*	75·4 (14)	73·5 (9)	74·6 (23)	75·4 (61)	74·4 (40)	75·6 (71)	75·7 (51)
100 *H′*/*L*	73·4 (13)	72·4 (8)	73·0 (21)	70·9 (61)	71·0 (33)	69·4 (43)	71·4 (34)
100 *B*/*H′*	103·3 (13)	102·7 (8)	103·1 (21)	106·3 (77)	105·5 (33)	109·9 (41)	104·4 (33)

Other series	Anglo-Saxon		Seventeenth-century London			Modern Scottish	
	London museums	Burwell	Farringdon Street	Whitechapel	Moorfields	Lowland	Glasgow
L	190·6 (58)	189·6 (45)	188·8 (139)	189·1 (137)	189·2 (44)	188·8 (54)	188·2 (524)
B	141·7 (103)	141·7 (45)	142·4 (141)	140·7 (135)	143·0 (46)	142·1 (54)	139·1 (524)
H′	136·0 (31)	136·3 (40)	129·7 (118)	132·0 (122)	129·8 (34)	133·6 (52)	132·9 (521)
100 *B*/*L*	74·7 (52)	74·8 (45)	75·4 (132)	74·3 (131)	75·5 (42)	75·3 (54)	75·0 (524)
100 *H′*/*L*	71·2 (25)	71·9 (40)	68·6 (115)	70·0 (120)	68·4 (31)	70·9 (52)	70·7 (521)
100 *B*/*H′*	104·9 (61)	104·3 (40)	109·8 (117)	106·6 (122)	110·2 (34)	106·4 (52)	104·9 (521)

representing the Iron Age population are obviously required, but the data available clearly focus attention on mean differences between the absolute and relative magnitudes of the calvarial height. It can be seen from the distribution on p. 301 above that all the male skulls from Maiden Castle have height-length indices greater than 69·0. In Table IV there are two seventeenth-century London series with means for the index less than this value, and one of the Romano-British series has a mean of 69·4. These conditions indicate unusual separation of the distributions for different series, and there is no reason to suspect that any cranial characters other than the height and the height indices would distinguish the populations represented in Table IV as effectively.

4. MEASUREMENTS OF THE MANDIBLES

Measurements of the mandibles were taken in accordance with the revised technique given in *Biometrika* (Morant *et al.* 1936, Appendix), and readings for individual bones are not provided in the present paper. It has been found (Cleaver, 1937) that larger numbers of lower jaws than of crania are needed to

give any decisive racial comparisons, and hence the means for all the adult specimens of Iron Age and Romano-British date taken together are the only ones worth considering. They are given in Table V for totals of thirty-four male and twenty-five female bones, and values for Anglo-Saxon (Morant, 1926) and a seventeenth-century London (Cleaver, 1937) series are included for comparative purposes.

TABLE V

Mean measurements of English series of mandibles

	Male			Female		
	Anglo-Saxon	Maiden Castle: Iron Age	Farringdon Street, London: seventeenth century	Anglo-Saxon	Maiden Castle: Iron Age	Farringdon Street, London: seventeenth century
w_1	123·7 (25)	121·5 (19)	117·7 ± 0·53 (23)	116·6 (22)	117·5 (12)	110·8 ± 0·64 (30)
g_0g_0	100·4 (33)	99·5 (25)	97·7 ± 0·70 (40)	92·9 (35)	90·8 (18)	85·7 ± 0·55 (50)
c_rc_r	100·3 (27)	99·3 (20)	95·9 ± 0·68 (29)	93·2 (28)	95·9 (18)	91·7 ± 0·54 (35)
zz	45·3 (57)	44·8 (33)	43·9 ± 0·24 (40)	44·1 (50)	42·5 (24)	42·9 ± 0·25 (49)
c_pl	21·7 (38)	21·0 (27)	19·8 ± 0·20 (34)	19·1 (35)	20·0 (14)	18·0 ± 0·16 (45)
ml	107·2 (31)	105·3 (24)	104·1 ± 0·64 (34)	104·2 (45)	98·9 (16)	99·4 ± 0·61 (43)
c_pl	77·2 (42)	79·0 (27)	74·9 ± 0·40 (40)	74·6 (49)	70·5 (19)	69·7 ± 0·37 (50)
rb'	33·2 (61)	33·1 (33)	30·9 ± 0·28 (40)	31·0 (56)	30·4 (24)	28·3 ± 0·25 (50)
m_2p_1	28·1 (59)	28·4 (23)	28·2 ± 0·18 (22)	27·6 (57)	27·2 (19)	27·7 ± 0·22 (19)
h_1	33·1 (40)	34·7 (22)	30·9 ± 0·46 (12)	30·5 (31)	30·5 (17)	29·7 ± 0·36 (26)
m_2h	27·2 (51)	28·2 (20)	24·9 ± 0·45 (19)	24·4 (52)	26·6 (15)	23·6 ± 0·51 (12)
c_rh	65·7 (48)	70·6 (28)	64·8 ± 0·47 (40)	59·2 (47)	60·2 (25)	56·5 ± 0·47 (50)
rl	64·0 (45)	66·2 (26)	62·2 ± 0·40 (36)	59·1 (45)	57·3 (20)	53·5 ± 0·44 (43)
$M\angle$	120°·3 (47)	116°·5 (27)	121°·7 ± 0·60 (40)	122°·5 (49)	123°·1 (19)	127°·8 ± 0·59 (50)
$R\angle$	72°·0 (36)	77°·6 (24)	72°·0 ± 0·94 (37)	68°·2 (36)	71°·6 (18)	71°·2 ± 0·87 (48)
$C'\angle$	68°·2 (32)	69°·8 (18)	61°·8 ± 1·35 (16)	70°·0 (25)	70°·3 (14)	67°·3 ± 0·80 (27)
$100\ c_rh/ml$	60·9 (27)	67·9 (22)	62·4 ± 0·66 (34)	58·3 (38)	60·5 (16)	56·8 ± 0·49 (43)
$100\ c_rc_r/ml$	94·4 (15)	93·9 (17)	92·3 ± 1·03 (25)	91·7 (26)	97·1 (12)	92·4 ± 0·74 (30)
$100\ g_0g_0/c_pl$	129·0 (32)	127·4 (25)	130·9 ± 1·23 (40)	126·2 (35)	129·2 (18)	123·3 ± 1·02 (50)
$100\ rb'/rl$	51·5 (45)	50·8 (26)	50·0 ± 0·54 (36)	53·0 (43)	53·5 (19)	53·2 ± 0·66 (43)
$100\ g_0g_0/c_rc_r$	99·3 (19)	102·1 (18)	102·9 ± 1·07 (29)	99·3 (23)	96·2 (14)	94·0 ± 0·86 (35)

None of these is long enough to give an adequate representation of the type for the population it represents, and probable errors are only available for the last. A few general comparisons are sufficiently suggestive, however, to be of interest. For both sexes the two earlier types are very similar in size, and for several characters both appear to be significantly larger than that of the Farringdon Street series. There is only one measurement of size which provides a clear exception to this rule—viz. the length of the dental arcade from second molar to first premolar (m_2p_1)—and it is the only measurement taken relating to the size of the teeth. Both Anglo-Saxon and Iron Age types appear to be distinguished from that of seventeenth-century Londoners on account of less protruding chins ($C'\angle$ greater), and the Iron Age appears to be distinguished

from the other two on account of a more outstanding coronoid process (judging from $R\angle$ and $100\, c_r h/ml$). Otherwise, there is no clear evidence of distinctions between the types.

Judging from all the characters, the Anglo-Saxon and Maiden Castle populations were just distinguished by features of their lower jaws, but the resemblance between them was closer than that between either and the later Londoners. The relations found favour the hypothesis that the mandibles of Englishmen, but not their teeth, became slightly smaller in historical times, but more data will be needed to substantiate it. If the hypothesis be accepted, then mandible measurements should not be used to estimate racial relationships unless allowance is made for secular changes in them within the same population.

5. MEASUREMENTS OF THE LENGTHS OF THE LONG BONES

Individual readings are given in Table VIII, and means of the adult series for the characters of greatest interest are in Table VI. The lengths were taken in the ways specified by Münter (1936), whose means for Anglo-Saxon skeletons are quoted. In spite of the small numbers, it was thought worth while computing separate means for (*a*) the Belgic War Cemetery series, and (*b*) for all the other specimens of Iron Age or Roman date from Maiden Castle.

For both sexes all the absolute measurements for these two series in Table VI are less than the corresponding values for Anglo-Saxons. A rough appreciation of the significance of the differences between the means can be obtained by supposing that the two Maiden Castle populations considered, and also that made up by combining them, exhibited the same variabilities as the total Anglo-Saxon population. This may appear to be a very arbitrary assumption, but in fact it is not unreasonable, since a close approach to equality in variation is usually found on comparing different subgroups with a total population, and also on comparing distinct populations.

By applying the Anglo-Saxon standard deviations, we reach the conclusion that there are no significant differences between the means of the absolute and indicial measurements in the case of the Belgic War Cemetery and the other series from the same site, and this is true for both sexes. On the same assumption with regard to variation, it is found that the pooled Maiden Castle means only show markedly significant differences from the Anglo-Saxon in the case of the length of the femur for males, the length of the humerus for males and females, and the radius-humerus index for males. The differences between the statures cannot be supposed clearly significant in the case of either sex, but it is safe to conclude from the evidence of both that the Iron Age inhabitants of Maiden Castle were shorter than Anglo-Saxons. The sex ratios for stature are almost identical, being 1·076 for the former and 1·073 for the latter series, and of the order usually found. In inches the estimates of height obtained are

TABLE VI

Means for lengths of the right femur, tibia, humerus and radius, and indices derived from these lengths, for Maiden Castle (Iron Age and Romano-British) and Anglo-Saxon series of adult skeletons

	Male			
	Other than Belgic War Cemetery	Belgic War Cemetery	All Maiden Castle	Anglo-Saxon
F. max.	437·6 (11)	443·2 (15)	440·8 (26)	463·3 ± 1·22 (153)
T. max.*	362·8 (11)	371·2 (18)	368·0 (29)	378·9 ± 1·46 (103)
H. max.	323·0 (12)	326·8 (17)	325·2 (29)	337·1 ± 1·15 (121)
R. max.	240·8 (10)	250·9 (15)	246·9 (25)	251·6 ± 1·12 (79)
100 T. obl./F. obl.	81·2 (10)	82·5 (14)	82·0 (24)	81·1 ± 0·17 (92)
100 H. max./F. obl.	73·9 (11)	74·3 (13)	74·1 (24)	73·5 ± 0·14 (100)
100 R. max./H. max.	75·4 (10)	77·2 (15)	76·4 (25)	74·6 ± 0·19 (62)
100 (H. max. + R. max.)/(F. obl. + T. max.)†	71·2 (9)	71·6 (12)	71·5 (21)	70·7 ± 0·16 (41)
Reconstructed stature	1640 (13)	1654 (19)	1649 (32)	1683 (161)‡
	Female			
F. max.	411·9 (13)	411·0 (8)	411·5 (21)	426·1 ± 2·05 (56)
T. max.*	336·4 (14)	342·7 (7)	338·5 (21)	350·4 ± 1·99 (44)
H. max.	300·8 (11)	296·4 (8)	298·9 (19)	312·5 ± 1·54 (47)
R. max.	223·0 (13)	215·1 (7)	220·3 (20)	227·6 ± 1·41 (34)
100 T. obl./F. obl.	81·5 (13)	81·2 (6)	81·4 (19)	80·8 ± 0·20 (38)
100 H. max./F. obl.	74·5 (8)	72·3 (7)	73·5 (15)	73·6 ± 0·25 (36)
100 R. max./H. max.	74·2 (9)	73·4 (7)	73·8 (16)	73·5 ± 0·22 (31)
100 (H. max. + R. max.)/(F. obl. + T. max.)†	70·8 (7)	69·7 (5)	70·3 (12)	70·6 ± 0·21 (21)
Reconstructed stature	1535 (17)	1527 (9)	1532 (26)	1568 (59)‡

* Maximum length of the tibia including the spine. † Maximum length of the tibia excluding the spine.

‡ The reconstructed statures for Anglo-Saxons were found from the mean lengths for different long bones, and the numbers of individuals given are the numbers of femora involved. The statures actually relate to rather larger numbers of skeletons.

5 ft. 6¼ in. for Anglo-Saxon and 5 ft. 5 in. for Iron Age men, and 5 ft. 1¼ in. for Anglo-Saxon and 5 ft. 0¼ in. for Iron Age women. The average stature of the general male adult population of England to-day is about 5 ft. 7¼ in., and for different social classes means between about 5 ft. 5¾ in. and 5 ft. 9¼ in. are found. The Iron Age men were thus decidedly short compared with modern Englishmen, and the estimated stature for them is very close to the average found for all European populations to-day.

For the measurements considered, the only distinction between the two ancient populations other than that in size is found for the index expressing the length of the radius as a percentage of the length of the humerus in the case of

the male, but not in that of the female, sample. It might be suggested that the proportions of the upper and fore-arms of the men who lived at Maiden Castle were influenced on the right, or on both, sides by continual practice in the use of the sling, which is known to have been one of their principal weapons from the evidence of caches of stones. The following means, which relate only to paired indices, are of interest in this connexion:

	100 × Radius max./Humerus max.				100 × (H. max. + R. max.)/(F. obl. + T. max. ex spine)			
	Maiden Castle		Anglo-Saxon		Maiden Castle		Anglo-Saxon	
	Male	Female	Male	Female	Male	Female	Male	Female
R.	76·3 (18)	73·6 (14)	74·6 (25)	74·0 (14)	71·5 (11)	70·3 (12)	70·9 (21)	70·4 (9)
L.	78·4 (18)	75·0 (14)	75·7 (25)	74·6 (14)	70·8 (11)	68·9 (12)	70·0 (21)	69·6 (9)
R.-L.	−2·1	−1·4	−1·1	−0·6	+0·7	+1·4	+0·9	+0·8

All the series are very restricted in size, but there is complete agreement in the signs of the side and sex differences found. The most significant difference between the two populations appears to be that for the male radius-humerus index on the left side. This might suggest that the Maiden Castle men were left-handed slingers, but their intermembral indices are so close to the Anglo-Saxon values that it seems unsafe to accept the hypothesis that the lengths of their arms were affected by use. More abundant material might tell definitely for or against such a view, and a detailed anatomical examination of the arm bones of the Maiden Castle and modern series would also be relevant to the question.

6. SUMMARY AND CONCLUSIONS

The material studied consists of the imperfect skeletal remains of eighty-three individuals, who are distributed in Table I according to periods, sexes, and ages at death. This paper is concerned chiefly with the customary measurements of the crania and mandibles, and with the lengths of the long bones. Comparisons of various kinds are made between:

A, the Belgic War Cemetery series, representing the defenders of the Castle who were massacred by Roman invaders in A.D. 43, and

B, a composite series made up by all the other specimens of Iron Age and Roman date.

There is no clear distinction between the distributions of the ages at deaths of the adults forming the two short series, but for both the men died at a rather younger age, on the average, than those interred in English cemeteries of later

date, while the females are not distinguished in the same way. Remarks on cranial anomalies are provided. The teeth of the inhabitants of Maiden Castle were not well preserved.

Judging from their appearance, nearly all the crania in series *A* and *B* would be considered quite unexceptional if found in any British collection of later date than the Bronze Age, but there are few examples of the markedly retreating frontal bone which characterizes seventeenth-century Londoners. There is agreement between the male and female samples and this is confirmed by measurements of the skulls and long bones.

There are no statistically significant differences between the mean cranial (Table III) and long bone (Table VI) measurements of series *A* and *B*. Both are characterized by a large calvarial height, and the estimates of stature they give are about 1 in. less than the Anglo-Saxon values. The type of the total series ($A+B$) of crania is found to be indistinguishable from the Anglo-Saxon, while both are differentiated from the types of seventeenth-century London series. This conflicts with the conclusion reached previously, from very inadequate data, for Iron Age and Romano-British crania, to the effect that the Iron Age and recent types are very similar while the Anglo-Saxon stands apart from both. More abundant material representing the Iron Age and Romano-British populations will be required to disclose the relationships of these closely allied groups. The cephalic index is practically constant for them, and the cranial types are distinguished most clearly by differences in the absolute and relative magnitude of the calvarial height (Table IV).

Measurements of the mandibles make a slight distinction between the Maiden Castle and Anglo-Saxon series, while both types are larger than that of seventeenth-century Londoners. The Iron Age men, but not the women, are distinguished from Anglo-Saxons by having a larger radius-humerus index. It is not clear that this difference is due to the fact that the men at Maiden Castle were slingers.

APPENDIX

Tables (VII and VIII) of individual measurements and remarks

The letters denoting periods (or groups) given in the third columns of the tables are: N. = Neolithic, I.A. = Iron Age (*A*, *B* or *C*), S. = Saxon, R.-B. = Romano-British, B. = Belgic, B.W.C. = Belgic War Cemetery. The letters denoting cranial measurements are those used in all craniometric papers in *Biometrika*. A list of them is given in the present volume, p. 162, but this does not include B'' = maximum frontal breadth (Martin, No. 10), and *Biast. B.* = biasterionic breadth (M. 12). Owing to their fragility, the capacities of the crania could not be determined by any direct method. The estimates given in Table VII were obtained by applying the reconstruction formulae involving *L*, *B* and H' given by Hooke (1926, p. 33). The reconstructed statures in Table VIII were obtained by applying the formulae given by Pearson (1898), using as many as possible of the bones involved in each case. The lengths of the long bones were determined in the ways adopted by Münter (1936). All the readings given in the two tables, whether queried or not, can be considered close approximations to the true values, with the possible exception of the few measurements of

the Neolithic skeletons enclosed in square brackets. These are more uncertain than the others, but they are given because the specimens are of particular interest.

Unless otherwise stated in the remarks given in Table VII, the cranium and mandible are complete, or almost complete, with both dental arcades intact and no teeth lost from either jaw before death, and the coronal, sagittal and lambdoid sutures are open. Absence of a third molar denotes that the tooth had never erupted, as far as can be told. Two male skulls are not in the tables, since no measurements can be given for them, but they were included in determining the frequencies of qualitative features. These are:

P 24 (B.W.C.). Sagittal suture closed, coronal and lambdoid beginning to close. Upper jaw incomplete and some teeth lost before death, apparently only 1 incisor R.; 2 teeth lost from mandible and M 3's absent, abscess cavities at roots of M 1's.

R 2 (? I.A.). Sagittal suture closing. Upper jaw missing. Superficial wound on frontal bone.

LIST OF PLATES

Plate I. A typical male skull (P 30) from the Belgic War Cemetery. This specimen has a cephalic index which is very close to the mean for the series, but its height-length index (70·9) is below the average (73·4) and it is peculiarly orthognathous ($P\angle = 90°$). The lower incisors are crowded.

Plate II. Exceptional skulls from the Belgic War Cemetery.

A. A female adolescent cranium (P 20) with injury to the right malar bone.

B. A male cranium (P 7 A) with a trace of a suture on the right parietal bone.

C and D. A female skull (P 36) with an exceptional type of facial skeleton and high cephalic index (87·0).

Plate III. Skulls from Maiden Castle with dental and other anomalies.

A. A female cranium (T 21) with cyst cavities at the roots of the second premolars.

B. The broken basi-occipital part of a male cranium (O 3) showing the extension into it of sphenoidal air sinuses.

C. The palate of a male cranium (P 7) with a large anterior palatine foramen.

D. The palate of a female cranium (T 28) showing absence of the canines and rotation of the first premolars.

E. A female mandible (P 14) with the left third molar impacted and mandibular torus.

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No.	Year excavated	Period	Sex	*L*	*B*	*B′*	*B″*	*Biast. B*	*H′*	S_1'	S_2'	S_3'	*S*
Q 1	1937	N.	♂	[195]	[140]	100·5	—	—	—	122·0?	118·3?	—	40
R 5	„	„	♂?	[195]	[139]	101·1?	—	109?	—	112·0?	124·0?	104·2?	26
Pit 3	1936	I.A.: *A*	♂	189	138	93·0	—	—	—	108·0	115·0	—	—
O 3	„	I.A.: *B*	„	—	—	89·0	—	118?	—	—	—	—	—
T 4	1937	I.A.: *C*	„	—	—	—	—	—	142	—	120·4?	98·8?	—
T 10	„	„	„	198·5?	133	100·4	—	—	—	116·2	—	—	31
T 11	„	„	„	189	142·5	92·2	122	113	138·5	114·1	123·7	102·3	29
T 20	„	„	„	193	139	101·1	122	110·5	140	117·8	123·2	98·4	30
T 25	„	„	♂?	176	130	94·1	115	103	133·5	111·1	110·3	95·0	30
N 1	1936	„	„	186?	141?	95·8	—	—	137?	112·0	107·6	101·1?	25
P 2	1937	I.A.: *C* (B.W.C.)	♂	—	—	—	—	119·5?	140·5?	108·2?	106·2?	103·5?	—
P 6	„	„	„	194?	135?	105·6	125	111	141	115·5	119·4?	97·8	34
P 7	„	„	„	188?	138?	97·1?	—	111?	137?	112·2?	124·8?	107·2?	28
P 7A	„	„	„	189	144	97·8	125·5	118	149·5	124·6?	120·5?	102·0?	43
P 9	„	„	„	186	144·5	98·0	117	121?	129·5?	—	—	99·2	—
P 11	„	„	„	187?	145·5	95·7	—	117?	137?	112·1?	115·0?	101·8	26
P 12	„	„	„	—	—	—	—	—	—	—	—	104·2?	—
P 18	„	„	„	181·5	140·5	96·8	125	119	139	109·9	112·0	101·2	23
P 19A	„	„	„	—	—	—	—	—	—	—	—	—	—
P 22	„	„	„	201?	144?	95·5	—	—	—	123·0	121·3	—	45
P 23	„	„	„	184	142·5	101·5	119	117	136	115·0	112·4?	96·8?	32
P 25	„	„	„	184·5?	137?	87·7?	—	—	137?	—	—	—	—
P 27	„	„	„	193·5	148	102·1	130·5	116?	138·5	122·1	119·6	99·1	47
P 28	„	„	„	178	143	94·1	123·5	112	128	109·9	111·7	95·8	27
P 29	„	„	„	178	136	95·5	119·5	112	134	112·8	115·2	91·1	27
P 30	„	„	„	190·5	143	100·0	122·5	117·5	135	109·9	121·9	95·2	24
P 34	„	„	„	193?	139?	93·0	—	112	138	116·0?	113·6?	101·2	34
O 1	1936	R.-B.	♂?	187	141·5	98·4	114·5	109	129·5	108·2	108·2	102·0	21
O 2	„	„	♂	198	149	95·8	130	114·5	141	120·2?	129·3	103·2	40
QR V	1937	„	„	193	141	99·8	124	112?	134	115·3	110·6?	100·9?	36
T 9	„	? I.A. or R.-B.	„	—	145?	106·3	122?	—	—	109·7	—	—	26
T 16	„	„	„	—	—	97·2	123·5	—	—	116·3?	—	—	33
T 1	„	I.A.: *B*	♀	176	132	94·0	115	100	135	111·9	108·0	96·0	26
T 13	„	„	„	182·5	141	98·5	—	101?	135·5	107·9	121·6	101·4	25
T 14	„	„	„	183·5	134·5	94·0	—	—	136?	116·8	—	—	35
T 18	„	„	„	182	135·5	93·2	—	—	140	109·5	114·8	102·7	23
T 24	„	„	„	169	127	93·1	109	103	124	100·4	109·0	86·0	14
Q 1	„	„	♀?	180	135	98·1	111	110	134	110·1	104·8	99·3	24
Pit Y	1934	„	♀	180	—	94·0	—	—	136	113·1?	113·6	96·5	29
O 4	1936	I.A.: *C* (B.)	„	185	132·5	97·3	117	104·5	136	—	111·6	96·7	—
N 2	„	„	„	183·5	143	96·2	119	—	134	114·2	117·1	92·0?	30
T 6	1937	„	„	174	130	93·0	111	110	125	104·1	106·5	92·1	18
T 12	„	„	„	182?	140·5	97·4?	—	—	126?	108·8?	—	—	27?
T 21	„	„	„	180	—	93·5	111·5	—	124	116·0	104·4	—	34
T 22	„	„	„	192?	138?	91·1	—	—	132	115·3	115·1?	104·0?	29
T 26	„	„	„	182	135	97·8?	—	110	—	—	—	—	—
T 27	„	„	„	171	138	94·3?	—	—	—	—	—	—	—
P 5	„	I.A.: *C* (B.W.C.)	„	180	131·5	94·2	115·5	112	134	105·7	114·2	96·0	21
P 14	„	„	„	188?	—	96·3	119·5	—	139·5?	111·8	118·4	—	31
P 19	„	„	„	185	137?	98·3	120	115	135	115·6	116·2	100·2	34
P 26	„	„	„	188	136?	87·1	116	110·5	139	118·3	118·0	97·9	36
P 31	„	„	„	169	127·5	90·1	108	104·5	126·5	105·0	106·9	92·2	20
P 36	„	„	„	169	147	87·0	115?	106·5	133	112·9	108·4	98·2	29
P 37	„	„	„	174	135?	94·8	118·5?	106?	134?	111·1	115·6	95·2?	30
T 28	„	I.A.: *C* or R.-B.	„	188	—	95·5	—	—	—	—	—	—	—
QR II	„	R.-B.	„	184·5?	136?	88·4	—	—	130·5	110·0?	118·4?	—	25
QR III	„	„	♀?	177	143	89·0	117?	—	—	107·3	103·0	—	25
QR IV	„	„	♀	177·5	132	89·1	112·5	107·5	128	99·8	111·0	101·9	13
P 20	„	I.A.: *C* (B.W.C.)	♀ Adol.	177·5?	148·5?	103·4	132	—	130·5?	112·9	109·1?	97·8?	29
T 29	„	I.A.: *C*	„	179·5	140·0	92·0	118	106	139·0	109·0?	118·2?	96·8	25

S_1'	S_2'	S_3'	S_1	S_2	S_3	S	U	$\beta Q'$	fml	fmb	LB	GL	$G'H$	J	GB
122·0?	118·3?	—	40?	128?	—	—	—	—	—	—	—	—	—	—	—
112·0?	124·0?	104·2?	26·5?	136·5?	127	390	—	—	—	—	—	—	—	—	—
108·0	115·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	32·8	27·0	—	—	—	—	96·3
—	120·4?	98·8?	—	134·5	125·5?	—	—	—	39·3	31·5	—	—	—	—	—
116·2	—	—	31·5	—	—	—	535	318	—	—	—	—	—	—	—
114·1	123·7	102·3	29	136	118	383	531	320	38·9	35·3	102·0	89·5	72·8	132·5	98·2
117·8	123·2	98·4	36·5	140	117	394	540	322	36·1	31·5	106·1	92·5?	77·1?	—	—
111·1	110·3	95·0	30	120·5	115	366	496	305	35·0	26·8	98·2	85·5	69·9	123?	91·8
112·0	107·6	101·1?	25·5?	119·5	123?	369?	522?	—	—	—	—	—	—	—	—
108·2?	106·2?	103·5?	—	116?	123?	—	—	—	39·1?	28·4	108·0?	—	—	—	—
115·5	119·4?	97·8	34·5	135?	120	390	543?	311?	38·6	36·7	109·7	97·2	76·8	140	105·7
112·2?	124·8?	107·2?	28·5?	143?	134·5?	407?	534?	320?	33·0?	30·7?	94·2?	—	—	—	89·8
124·6?	120·5?	102·0?	43?	133·5?	121?	398	534	336	39·7	34·8	103·0	94·3	69·7	138	92·0
—	—	99·2	—	—	125·5	—	530	309	36·8	28·4	99·1	97·6	72·1	—	98·3
112·1?	115·0?	101·8	26?	127?	121?	374	—	—	—	—	103·0?	—	—	—	94·6
—	—	104·2?	—	—	128?	—	—	—	—	—	—	—	—	—	—
109·9	112·0	101·2	23·5	128	123·5	375	521	313	39·8	32·8	106·2	94·1	75·0	—	93·5
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
123·0	121·3	—	45·5	135	—	—	—	332	—	—	—	—	—	—	—
115·0	112·4?	96·8?	32·5	124·5	118·5	375	521	317	36·7	29·3	106·0	99·4	74·7	134	94·3
—	—	—	—	—	—	—	508?	307	36·6?	—	—	—	—	—	—
122·1	119·6	99·1	47·5	131?	126	404?	554	332	34·1	29·9	103·3	95·4	69·3	137·5	98·8
109·9	111·7	95·8	27·5	126	117	371	518	309	39·5	30·7	92·1	—	—	—	—
112·8	115·2	91·1	27	130	113	370	508	306	35·8	29·9	98·5	—	—	—	104·9
109·9	121·9	95·2	24·5	135	121·5	382	537?	310?	39·1	33·7	103·1	89·9	67·2	137·5	95·6
116·0?	113·6?	101·2	34?	127·5?	123	384	535?	312	43·1	34·3	106·5	96·4	80·6	139	94·1
108·2	108·2	102·0	21·5	120·0	127	370	523	306	35·6	30·2	99·2	89·0	71·0	123	89·6
120·2?	129·3	103·2	40·0	146·0	121	407	559	335?	40·5	35·9	105·0	91·4?	74·8	133	86·0
115·3	110·6?	100·9?	36·0	120?	129?	384	541?	313	39·9	31·7	102·5	95·5	62·5	132	91·1
109·7	—	—	26·5	—	—	—	—	—	—	—	—	—	—	—	98·1
116·3?	—	—	33?	—	—	—	—	—	—	—	—	—	—	—	—
111·9	108·0	96·0	26	123·5	112	362	491?	302	35·1	26·4	97·8	—	—	—	83·1
107·9	121·6	101·4	25·5	139	121	385	523	320	33·5	—	100·2	—	—	121	80·0
116·8	—	—	35·5	—	—	—	516?	302?	—	—	—	—	70·1	—	88·0
109·5	114·8	102·7	23·5	125	121	370	510	312	34·9	31·0	103·0	—	—	—	—
100·4	109·0	86·0	14·5	125·5	100	440	478	285	33·0	29·0	94·4	91·4	59·9	120	89·0
110·1	104·8	99·3	24	115·5	120	359	506	298	34·8	28·1	104·2	99·5	63·0	124	92·1
113·1?	113·6	96·5	29	129	114	371	—	—	32·8	26·8	102·3	—	—	—	—
—	111·6	96·7	—	123	116	—	510?	306	33·4	27·8	—	—	—	—	—
114·2	117·1	92·0?	30·	130	112?	373·5	521?	321	37·0	28·5	101·7	—	—	—	84·1
104·1	106·5	92·1	18·5	122·5	111	352	495	292	37·8	30·8	97·0	87·0?	68·5?	120?	93·1
108·8?	—	—	27?	—	—	—	—	—	39·5	—	—	—	—	—	—
116·0	104·4	—	34·5	111	—	—	510	303	—	—	99·9	101·8	75·1	124·5	—
113·3	115·1?	104·0?	29·5	132?	126?	388?	536?	305?	—	—	101·1	—	69·2	—	93·1
—	—	—	—	—	—	—	511	301	—	27·8	101·2	92·8	69·3	127	—
—	—	—	—	—	—	—	507?	327	—	—	—	—	—	—	—
105·7	114·2	96·0	21·5	127·5	114	363	506	307	—	—	103·1?	—	63·6?	121?	—
111·8	118·4	—	31·5	132·5	—	—	—	316?	—	—	98·1?	85·4?	65·3	—	86·0
115·6	116·2	100·2	34·5	131	122	387	523?	308?	38·3	33·3	98·1	83·3	69·0	127·5	91·1
118·3	118·0	97·9	36	132·5	120	389	518	310?	39·9	29·1	101·7	85·3	74·0	126	97·1
105·0	106·9	92·2	20	117	113	349	477	281	34·7	24·7	90·0	—	—	—	—
112·9	108·4	98·2	29·5	121	118	369	499	318	27·7	24·9	94·0	91·2	67·7	—	91·0
111·1	115·6	95·2?	30·5	132	109?	371	501?	306	38·7?	29·0?	92·1?	86·3?	68·8?	—	92·0
—	—	—	—	—	—	—	—	—	—	—	—	—	72·1?	—	—
110·0?	118·4?	—	25?	131?	—	—	—	307?	—	—	—	—	—	120	—
107·3	103·0	—	25	112·5	—	—	507	303	—	—	—	—	68·0?	—	—
99·8	111·0	101·9	13·5	119	123	355·5	501	292	36·2	32·5	95·8	89·3	62·0	124·5	89·6
112·9	109·1?	97·8?	29	122·5?	115·5?	367	520?	321?	40·9	31·6	92·6?	—	—	—	—
109·0?	118·2?	96·8	25?	132?	118?	374	512	319	34·0	31·8	98·1	85·8	70·0	—	—

TABLE VII

Measurements of crania from Maiden Castle

H	J	GB	NH, L	NB	O_1L	O_2L	G_1'	G_2	SC	SS	C	$100\frac{B}{L}$	$100\frac{H'}{L}$	$100\frac{B}{H'}$	Oc.I.	$100\frac{fmb}{fml}$	$100\frac{G'H}{GB}$	$100\frac{NB}{NH}$, L	10
—	—	—	—	—	—	—	—	—	—	—	—	[71·8]	—	—	—	—	—	—	
—	—	—	—	—	—	—	—	—	9·9	—	—	[71·3]	—	—	58·7?	—	—	—	
—	—	—	—	—	—	—	—	—	—	—	—	73·0	74·3?	98·2?	—	—	—	—	
—	—	96·3	—	23·9	—	—	—	41·1	—	—	—	—	—	—	—	82·3	—	—	
—	—	—	—	—	—	—	43·9	—	—	—	—	—	—	—	56·2?	80·2	—	—	
—	—	—	—	25·8	—	—	43·1	38·1?	—	—	—	67·0?	—	—	—	—	—	—	
8	132·5	98·2	55·0	24·5	40·6	34·0	42·0	—	9·8	5·6	1564	75·4	73·0	103·3	64·6	90·7	74·1	44·5	
1?	—	—	56·7?	—	—	—	45·5	40·0	12·0	5·9	1574	72·0	72·5	99·3	60·9	87·3	—	—	
9	123?	91·8	50·5	23·8	39·5	32·8	41·1	—	8·3	3·8	1317	73·9	75·9	97·4	59·2	76·6	76·1	47·1	
—	—	—	—	—	—	—	—	—	—	—	1514?	75·8?	73·7?	102·9	58·8?	—	—	—	
—	—	—	—	—	—	—	—	40·4	—	—	—	—	—	—	61·0?	72·6?	—	—	
8	140	105·7	53·6	25·4?	45·8	32·6	49·2	—	14·5	3·6	1551?	69·6?	72·7?	95·7?	58·2	95·1	72·7	47·4?	
—	—	89·8?	—	—	—	—	—	43·2?	—	—	1500	73·4?	72·9?	100·7?	56·8?	93·0?	—	—	
7	138	92·0	51·0	25·7	41·7	32·5	49·3	—	9·3	4·4	1688	76·2	79·1	96·3	61·1?	87·7	75·8	50·4	
1	—	98·3	47·8	—	40·7	28·7	—	—	9·6	5·2	1473	77·7	69·6?	111·6?	56·4	77·2	73·3	—	
—	—	94·6?	—	22·2?	—	—	49·3	40·8	—	—	1563?	77·8?	73·3?	106·2?	60·9?	—	—	—	
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	58·2?	—	—	—	
0	—	93·1	57·7	23·9	42·9	33·7	45·5	40·0?	14·9	6·0	1496	77·4	76·6	101·1	58·6	82·4	80·6	41·4	
—	—	—	—	—	—	—	43·5?	39·7?	—	—	—	—	—	—	—	—	—	—	
—	—	—	—	—	—	—	—	—	—	—	—	71·6?	—	—	—	—	—	—	
7	134	94·3	50·4	26·8	44·3	33·1	48·0	38·7	9·2	4·5	1504	77·4	73·9	104·8	58·4	79·8	79·2	53·2	
—	—	—	—	—	—	—	—	—	—	—	—	74·3?	74·3?	100·0?	—	—	—	—	
3	137·5	98·8	48·8	25·9	43·7	33·6	46·7	—	5·6	3·2	1651	76·5	71·6	106·9	56·2	87·7	70·1	53·1	
—	—	—	—	—	—	—	—	39·0	—	—	1391	80·3	71·9	111·7	58·6	77·7	—	—	
—	—	104·9	57·8	—	40·8	35·1	—	—	9·0	4·3	1386	76·4	75·3	101·5	57·5	83·5	—	—	
2	137·5	95·0	50·7	23·1	44·6	36·3	43·5	40·8	9·1	4·8	1545	75·1	70·9	105·9	56·0	86·2	70·7	45·6	
6	139	94·3	54·2	26·8	47·9	33·4	45·2	41·1	9·6	3·1	1554?	72·0?	71·5	100·7?	59·0	79·6	85·5	49·4	
0	123	89·0	47·8	23·9	40·0	30·4	46·2	40·5	8·9	5·6	1453	75·7	69·3	109·3	57·3	84·8	79·8	50·0	
8	133	86·9	51·4	22·9	42·6	34·6	47·2	43·6	10·5	6·1	1522	75·3	71·2	105·7	62·4	88·6	86·1	44·6	
5	132	91·2	50·1	28·6	43·0	31·0	46·3	45·0	9·8	6·1	1534	73·1	69·4	105·2	55·9?	79·4	68·5	57·1	
—	—	98·5	46·9	27·2	41·3	31·0	47·1	39·4	12·1	2·5	—	—	—	—	—	—	—	58·0	
—	—	—	—	—	—	—	44·0?	—	—	—	—	—	—	—	—	—	—	—	
—	—	83·5	48·5	21·5	41·3	31·4	40·6?	—	10·0	6·1	1347	75·0	76·7	97·8	63·0	75·2	—	44·3	
—	121	80·3	48·4	24·1	40·6	32·9	—	—	11·4	4·6	1475	77·3	74·2	104·1	60·5	—	—	49·8	
1	—	88·4	53·0	25·5	39·2	31·8	47·2	36·9	9·1	5·7	1428?	73·3	74·1?	98·9?	—	—	79·3	48·1	
—	—	—	51·1	—	—	—	—	—	9·9	6·2	1463	74·5	76·9	96·8	61·9	88·8	—	—	
9	120	89·3	45·1	27·8	38·9	30·5	45·0	38·5	8·5	3·1	1173	75·1	73·4	102·4	63·4	87·9	67·1	61·6	
0	124	92·3	44·4	22·6	43·0	31·3	47·6	41·2	10·9	4·8	1391	75·0	74·4	100·7	59·4	80·7	68·3	50·9	
—	—	—	—	—	—	—	45·2	—	—	—	—	—	75·6	—	61·6	81·7	—	—	
—	—	—	—	—	—	—	—	—	—	—	1419	71·6	73·5	97·4	60·0	83·2	—	—	
—	—	84·1	48·5	—	44·6	35·8	—	—	9·3	4·3	1486	77·9	73·0	106·7	58·8?	77·0	—	—	
8·5?	120?	93·5	—	—	—	—	39·5?	35·3	—	—	1234	74·7	71·8	104·0	59·6	81·5	73·3?	—	
—	—	—	—	—	—	—	—	—	—	—	1378?	77·2	69·2?	111·5?	—	—	—	—	
5·1	124·5	—	52·4	23·8	43·2	32·7	—	37·2?	8·2	4·3	—	—	68·9	—	—	—	—	45·4	
9·2	—	93·0?	46·9	—	41·2	31·8?	46·4	44·0	—	—	1479?	71·9?	68·8?	104·5?	59·2?	—	74·4?	—	
9·3	127	—	50·5	26·4	—	—	—	40·4	9·2	3·3	—	74·2	—	—	—	—	—	52·3	
—	—	—	—	—	—	—	—	—	—	—	—	80·7	—	—	—	—	—	—	
3·6?	121?	—	47·1?	—	—	—	41·0	—	—	—	1360	73·1	74·4	98·1	61·0	—	—	—	
5·3	—	86·1	44·6	20·2?	40·2?	33·2?	41·6	33·6	9·2	2·7	—	—	74·2	—	—	—	75·8	45·3?	
9·0	127·5	91·8	49·5	22·4	39·9	32·7	—	—	10·9	4·4	1451?	74·1?	73·0	101·5?	58·8	86·9	75·2	45·3	
4·0	126	97·0	50·0	23·7	42·7	33·7	—	40·2	—	—	1500?	72·3?	73·9	97·8?	58·3	72·9	76·3	47·4	
—	—	—	—	—	—	—	—	—	—	—	1197	75·4	74·9	100·8	58·3	71·2	—	—	
7·7	—	91·0	43·1	24·5	39·3	29·8	48·7	—	12·8?	—	1408	87·0	78·7	110·5	59·9	89·9	74·4	56·8	
8·8?	—	92·9	45·9	21·0?	44·3	33·0	44·5	—	7·7	4·0	1351?	77·6?	77·0?	100·7?	65·7?	74·9?	74·1?	45·8?	
2·1?	—	—	52·1	—	—	—	—	42·0	—	—	—	—	—	—	—	—	—	—	
—	120	—	—	—	—	—	—	—	—	—	1397?	73·7?	70·7?	104·2?	—	—	—	—	
8·0?	—	—	53·0	—	—	—	—	—	5·5	2·8	—	80·8	—	—	—	—	—	—	
2·0	124·5	89·0	47·4	22·6	40·1	29·0	40·8	39·3	6·8	3·6	1297	74·4	72·1	103·1	59·5	89·8	69·7	47·7	
—	—	—	—	—	—	—	—	40·3	—	—	1458?	83·7?	73·5?	113·8?	61·6?	77·3	—	—	
0·0	—	—	51·0	21·0?	—	—	41·9?	39·0	—	—	1477	78·0	77·4	100·7	58·7	93·5	—	41·2?	

$100\frac{fmb}{fml}$	$100\frac{G'H}{GB}$	$100\frac{NB}{NH}, L$	$100\frac{O_2}{O_1}, L$	$100\frac{G_2}{G_1'}$	$100\frac{SS}{SC}$	$P\angle$	$N\angle$	$A\angle$	$B\angle$	No.	
—	—	—	—	—	—	—	—	—	—	Q 1	Sagittal suture closing. Cranium resto
—	—	—	—	—	—	—	—	—	—	R 5	Incomplete skull-cap only; restored fr
—	—	—	—	—	—	—	—	—	—	Pit 3	Upper jaw defective but probably no
82·3	—	—	—	—	—	—	—	—	—	O 3	Sagittal suture closed, coronal and la extension of sphenoidal air sinuses.
80·2	—	—	—	—	—	—	—	—	—	T 4	Sagittal suture partly obliterated, cor
—	—	—	—	88·4?	—	—	—	—	—	T 10	All sutures nearly obliterated. Some t
90·7	74·1	44·5	83·7	—	57·1	86°·5?	59°	77°	44°	T 11	Sagittal suture closed, coronal beginni
87·3	—	—	—	87·9	49·2	—	58°	77°	45°	T 20	Sagittal and coronal sutures beginning
76·6	76·1	47·1	83·0	—	45·8	—	58°	78°	44°	T 25	Sagittal suture beginning to close. Mo
—	—	—	—	—	—	—	—	—	—	N 1	Sagittal suture beginning to close. He
72·6?	—	—	—	—	—	—	—	—	—	P 2	Sagittal suture nearly obliterated, cor
95·1	72·7	47·4?	71·2	—	24·8	87°	60°	77°	43°	P 6	Coronal suture beginning to close. So
93·0?	—	—	—	—	—	—	—	—	—	P 7	Sagittal suture closing. Probably no te palatine foramen
87·7	75·8	50·4	77·9	—	47·3	85°	63°	76°	41°	P 7A	Two upper molars lost before death. T
77·2	73·3	—	70·5	—	54·2	85°	67°	70°	43°	P 9	Coronal and sagittal sutures beginning
—	—	—	—	82·8	—	—	—	—	—	P 11	Posthumously distorted to some exte
—	—	—	—	—	—	—	—	—	—	P 12	Largely defective. Palate defective w abscess cavities
82·4	80·6	41·4	78·6	87·9?	40·3	87°·5	59°·5	77°	43°·5	P 18	Nasal bones fused
—	—	—	—	91·3?	—	—	—	—	—	P 19A	Coronal suture closed, sagittal nearly wound. Upper M 3's absent
—	—	—	—	—	—	—	—	—	—	P 22	Sagittal suture closing, coronal and la
79·8	79·2	53·2	74·7	80·6	48·9	86°?	64°	74°	42°	P 23	Sagittal suture obliterated, coronal beg
—	—	—	—	—	—	—	—	—	—	P 25	Sagittal and lambdoid sutures obliter
87·7	70·1	53·1	76·9	—	57·1	88°·5?	63°·5	76°	40°·5	P 27	Sagittal suture closed, coronal and la wound on frontal bone
77·7	—	—	—	—	—	—	—	—	—	P 28	Sagittal suture closed, coronal and la
83·5	—	—	86·0	—	47·8	—	—	—	—	P 29	Sagittal and coronal sutures beginnin
86·2	70·7	45·6	81·4	93·8	52·7	90°	59°	81°	40°	P 30	Sagittal and coronal sutures beginnin
79·6	85·5	49·4	69·7	90·9	32·3	85°	60°	73°·5	46°·5	P 34	Abscess cavities at roots of upper M
84·8	79·8	50·0	76·0	87·7	62·9	86°·5	60°	76°	44°	O 1	Coronal and sagittal sutures beginnin
88·6	86·1	44·6	81·2	92·4	58·1	90°	58°	78°	44°	O 2	Sagittal suture beginning to close. 1
79·4	68·5	57·1	72·1	97·2	62·3	81°?	65°·5	78°	36°·5	QR V	Upper M 3, L and both lower M 3's a
—	—	58·0	75·1	83·7	20·7	—	—	—	—	T 9	Sagittal suture nearly obliterated. Man
—	—	—	—	—	—	—	—	—	—	T 16	Upper jaw defective and 1 molar lost
75·2	—	44·3	76·0	—	61·0	—	—	—	—	T 1	Sagittal suture closed, coronal and la
—	—	49·8	81·0	—	40·4	—	—	—	—	T 13	Metopic suture beginning to close, co
—	79·3	48·1	81·1	78·2	62·6	86°?	—	—	—	T 14	Sagittal suture obliterated, coronal cl
88·8	—	—	—	—	62·6	—	—	—	—	T 18	Sagittal suture nearly obliterated, la
87·9	67·1	61·6	78·4	85·6	36·5	80°	69°	74°	37°	T 24	
80·7	68·3	50·9	72·8	86·6	44·0	80°	68°	76°	36°	Q 1	Malar bone divided by horizontal sut
81·7	—	—	—	—	—	—	—	—	—	Pit Y	No mandible. Probably no teeth lost
83·2	—	—	—	—	—	—	—	—	—	O 4	Sagittal and coronal suture closing, palate and 2 lost from mandible
77·0	—	—	80·3	—	46·2	—	—	—	—	N 2	Coronal suture beginning to close. A
81·5	73·3?	—	—	89·4?	—	87°?	60°·5?	76°?	43°·5?	T 6	Sagittal and coronal sutures closed,
—	—	—	—	—	—	—	—	—	—	T 12	R. side of palate only, from which no
—	—	45·4	75·7	—	52·4	85°	69°	67°	44°	T 21	Pterion in K, R. and L. Lower M 3'
—	74·4?	—	77·2?	94·8	—	83°	—	—	—	T 22	Depressed wound on R. side of front
—	—	52·3	—	—	35·9	87°	63°	76°	41°	T 26	Sagittal suture obliterated, coronal n
—	—	—	—	—	—	—	—	—	—	T 27	Sagittal and lambdoid sutures oblite
—	—	—	—	—	—	—	—	—	—	P 5	All molars and 1 incisor lost from u at root of I·1, R
—	75·8	45·3?	82·6?	80·8	29·3	—	59°?	80°?	41°?	P 14	Upper M 3's absent; lower M 3, L in
86·9	75·2	45·3	82·0	—	40·4	89°	56°·5	80°	43°·5	P 19	Coronal suture beginning to close. M
72·9	76·3	47·4	78·9	—	—	94°?	55°·5	79°	45°·5	P 26	Two teeth lost from upper jaw befor
71·2	—	—	—	—	—	—	—	—	—	P 31	Sagittal and coronal suture closing, l and most teeth lost
89·9	74·4	56·8	75·8	—	—	84°	66°·5	71°	42°·5	P 36	Coronal suture beginning to close. S
74·9?	74·1?	45·8?	74·5	—	51·9	—	63°?	72°?	45°?	P 37	Sagittal and coronal sutures beginn posthumous distortion
—	—	—	—	—	—	—	—	—	—	T 28	Sagittal and coronal sutures closing.
—	—	—	—	—	—	86°	—	—	—	QR II	Sagittal and coronal sutures beginnir Large anterior palatine foramen
—	—	—	—	—	50·9	—	—	—	—	QR III	Sagittal and coronal sutures beginni
89·8	69·7	47·7	72·3	96·3	52·9	81°·5	65°	76°	39°	QR IV	
77·3	—	—	—	—	—	—	—	—	—	P 20	Basal suture obliterated but M 3's n
93·5	—	41·2?	—	93·1?	—	88°	58°·5	77°·5	44°	T 29	M 3's not fully erupted

Remarks
g. Cranium restored from a considerable number of fragments and defective owing to mutilation
only; restored from a considerable number of fragments
but probably no teeth lost before death
d, coronal and lambdoid closing. Blunt spine projecting horizontally anterior to opisthion. Basi-occipital broken exposing
idal air sinuses. 1 upper incisor lost before death
y obliterated, coronal and lambdoid closed. Several teeth lost from upper jaw before death; mandible defective
literated. Some teeth lost before death from both jaws
l, coronal beginning to close. Some teeth lost before death from both jaws; all M 3's absent
sutures beginning to close. Fronto-temporal articulation R. and L. Some teeth lost before death from both jaws
ning to close. Most teeth lost before death from upper jaw but none from mandible
ning to close. Healed wound on right parietal
y obliterated, coronal and lambdoid partly obliterated. Upper M 3's probably lost before death
ning to close. Some teeth lost before death from both jaws
g. Probably no teeth lost before death and lower M 3's probably absent. Abscess cavity at root of lower PM 2, R. Large anterior
st before death. Trace of suture (33 mm.) on right parietal
sutures beginning to close. 6 teeth lost before death from upper jaw
ted to some extent
Palate defective with several teeth lost before death and 2 abscess cavities; 4 teeth lost before death from mandible and 2
d, sagittal nearly obliterated, lambdoid closing. Marked posthumous deformation. Depression above left orbit probably due to
's absent
ng, coronal and lambdoid beginning to close. Both jaws incomplete, several teeth lost before death
rated, coronal beginning to close: no distortion apparent. Milk molar retained in place of lower left second premolar which is missing
d sutures obliterated, coronal closing. Palate incomplete and several teeth lost before death; mandible complete and 4 teeth lost
d, coronal and lambdoid closing; occipito-mastoid sutures obliterated. 4 molars lost before death from each jaw. Small healed
one
d, coronal and lambdoid beginning to close
sutures beginning to close. Nearly all teeth lost from both jaws before death
sutures beginning to close. Upper M 3, R absent, lower incisors crowded
roots of upper M 1, R and M 2, R
sutures beginning to close. Lower M 3, L absent
nning to close. 1 upper molar lost before death
oth lower M 3's absent
y obliterated. Mandible defective but probably no teeth lost before death. Swelling on right maxilla below orbit, possibly due to tumour
and 1 molar lost before death
ed, coronal and lambdoid closing. Several teeth lost from upper jaw before death. Slight wound on frontal bone
nning to close, coronal and sagittal closing. Most teeth lost from both jaws before death
erated, coronal closing. 1 upper molar lost before death
ly obliterated, lambdoid closed, coronal closing. Most teeth lost from both jaws before death
—
by horizontal suture R. and L.
ably no teeth lost from upper jaw before death
l suture closing, lambdoid beginning to close. Button exostosis on frontal bone. All teeth lost before death from incomplete
rom mandible
nning to close. All teeth lost before death except 2 in mandible
l sutures closed, coronal closing. 2 teeth lost from upper jaw before death; lower M 3's absent
ly, from which no teeth had been lost before death
d L. Lower M 3's absent. Cyst cavities in buccal surfaces of maxillae: see Plate III A
n R. side of frontal bone above pterion
erated, coronal nearly obliterated, lambdoid closed. 3 teeth lost from upper jaw before death and 1 from mandible
oid sutures obliterated, coronal partly obliterated. Palate defective; several teeth lost from mandible before death
icisor lost from upper jaw before death and disastema between central incisors; 1 molar lost from mandible and abscess cavity
; lower M 3, L impacted but eruption nearly complete. Mandibular torus R. and L. Cranium slightly distorted posthumously
nning to close. Most molars lost before death and abscess cavities in both jaws
upper jaw before death; lower M 3, L absent and apparently only 3 incisors erupted
l suture closing, lambdoid beginning to close. Palate incomplete and most teeth lost before death; lower dental arcade complete
t
nning to close. Several teeth lost from both jaws before death. Facial skeleton of unusual form: see Plate II, C and D
al sutures beginning to close. Several teeth lost from both jaws before death; abscess cavity at root of lower M 2, R. Slight
tion
l sutures closing. Upper canines apparently never erupted and first premolars rotated: see Plate III D
l sutures beginning to close. 2 upper and 1 lower molar lost before death; cyst cavity in buccal surface of mandible below M 2 R.
atine foramen
l sutures beginning to close. 3 upper molars lost before death
—
rated but M 3's not fully erupted. Lower incisors crowded. Malar bone R. deformed
pted

No.	Year excavated	Period	Sex	Femora				Tibiae					
				Maximum		Oblique		Maximum		Max. ex spine		Oblique	
				R.	L.	R.	L.	R.	L.	R.	L.	R.	L.
Q 1	1937	N.	♂	—	[442]	—	—	—	—	[370]	372?	—	—
Pit 3	1936	I.A.: *A*	,,	—	—	—	—	—	—	—	—	—	—
O 3	,,	I.A.: *B*	,,	399·5?	—	397·5?	—	—	—	—	—	—	—
T 3	1937	I.A.: *C*	,,	—	—	—	—	—	352	—	346	—	345
T 4	,,	,,	,,	454	458	451	454	367	368?	362	363	361	361
T 5	,,	,,	,,	—	—	—	—	377	—	372	—	372	—
T 10	,,	,,	,,	458	462	456	460	381	—	376	—	374	—
T 11	,,	,,	,,	439	442	437	440	372	371	369	368	368	366
T 20	,,	,,	,,	435	437	432	434	351?	350	347?	346	343	342
T 25	,,	,,	♂?	420	—	417	—	351	—	348	—	347	—
N 1	1936	,,	,,	454·5	457·5	452?	452·5	358·5	359·5	355·5	355·5	354·5	355
P 2	1937	I.A.: *C* (B.W.C.)	♂	452	452	449	451	381	383	378	379	376	377
P 6	,,	,,	,,	452	—	—	—	376	379	373	376	371	372
P 7	,,	,,	,,	453	456	450	454	386	386	381	383	380	380
P 7A	,,	,,	,,	451	460	446	457	382	—	379	—	377	—
P 8	,,	,,	,,	—	460	—	456	—	371	—	368	—	367
P 9	,,	,,	,,	429	430	426	426	338	—	335	—	335	—
P 11	,,	,,	,,	453?	448?	449?	446?	367	367?	363	363?	362	359
P 12	,,	,,	,,	—	—	—	—	384	384	380	380	379	377
P 18	,,	,,	,,	—	495	—	491	412	414	407	411	406	404
P 19A	,,	,,	,,	401	405	397	401	329	329	324	324	324	322
P 22	,,	,,	,,	—	431	—	429	356?	—	353?	—	351?	—
P 23	,,	,,	,,	458?	459	453?	456	377	378?	374	375?	373	372
P 24	,,	,,	,,	451?	459	450?	456	375	381	371	377	371	372
P 25	,,	,,	,,	474?	477	470?	475	398	—	393	—	392	—
P 27	,,	,,	,,	419	422	417	420	353	357	348	352	347	350
P 28	,,	,,	,,	411	420	407	416	326?	330	321?	322	320?	324
P 29	,,	,,	,,	416	420	414	419	359	357	352	353	351	350
P 30	,,	,,	,,	460	462	457	459	398	397	394	394	393	392
P 34	,,	,,	,,	468	466	466	463	385	389	383	386	380	380
O 1	1936	R.-B.	♂?	445?	449·5	443?	446	364·5?	366·5	361·5?	362	360?	360
O 2	,,	,,	♂	421?	—	418·5?	—	348?	—	347?	—	343·5?	—
QR V	1937	,,	,,	444	445	438	439	355	362	354	360	352	352
T 16	,,	? I.A. or R.-B.	,,	444	451	442	446	366	373?	363	370?	359	359
Q	,,	S.	,,	498?	497?	496?	492?	—	—	—	—	—	—
T 1	,,	I.A.: *B*	♀	409	408	406	404	333	—	330	—	329	—
T 13	,,	,,	,,	—	401	—	396	322?	324	319?	321	318?	317
T 14	,,	,,	,,	—	423	—	419	—	—	—	—	—	—
T 18	,,	,,	,,	—	—	—	—	—	353	—	349	—	348
T 24	,,	,,	,,	380	373	376	370	322	318	319	315	319	316
Q 1	,,	,,	♀?	388	391	385	388	327	325	323	322	322	320
O 4	1936	I.A.: *C* (B.)	,,	389	394·5	387	392	323·5	321·5	321	318	319	317
N 2	,,	,,	♀	400·5	414·5?	394·5	400?	330·5	337	330	334·5?	326·5	330
T 6	1937	,,	,,	414	421	413	418	337	335	334	331	333	331
T 12	,,	,,	,,	401	—	397	—	324	326	320	322	317	320
T 21	,,	,,	,,	432	430	428	427	350	348	347	343	346	343
T 22	,,	,,	,,	432	442?	428	434?	351	356	347	351	346	350
T 26	,,	,,	,,	406	—	402	—	334	335	332	333	331	332
T 27	,,	,,	,,	—	—	—	—	—	—	—	—	—	—
P 5	,,	I.A.: *C* (B.W.C.)	,,	—	405	—	402	341	335	337	329	335	329
P 14	,,	,,	,,	455	457	450	453	373	374	368	370	367	368
P 16	,,	,,	,,	373	379?	370	374?	294	298	289	293	288	292
P 19	,,	,,	,,	457	456	454	453	357	358	352	353	350	350
P 26	,,	,,	,,	445	449	436	440	379	386	376	382	374	378
P 31	,,	,,	,,	365?	370	363?	365	—	302?	—	301?	—	301
P 33	,,	,,	,,	402	399	396	394	—	327	—	324	—	324
P 36	,,	,,	,,	368	372	365	369	316	304	312	301	311	299
P 37	,,	,,	,,	423	428?	419	425?	339?	343	337?	339	333	339
T 28	,,	I.A.: *C* or R.B.	,,	442	444	439	438	361	360	357	357	355	355
QR II	,,	R.-B.	♀?	444	441	441	439	366	363	361	358	361	358
QR III	,,	,,	♀?	—	—	—	—	—	—	—	—	—	—
QR IV	,,	,,	♀	417	417	414	412	329	330	326	325	324	325
T 17	,,	I.A.: *B*	♂?, Adol.	—	—	—	—	363	365	359	360	358	360
P 20	,,	I.A.: *C* (B.W.C.)	♀, Adol.	438	440	436	434	347	349	345	346	342	345

TABL

Measurements of long bones o

Tibiae				Fibulae		Humeri				Radii		Ulnae		Clavicles		100
Max. ex spine		Oblique		Maximum		Maximum		Oblique		Maximum		Maximum		Maximum		
R.	L.	R.	L.	R.	L.	R.	L.	R.	L.	R.	L.	R.	L.	R.	L.	I
[370]	372?	—	—	—	—	[330]	—	—	—	252?	248?	—	—	—	—	[7
—	—	—	—	—	—	339?	319	—	313·5	—	251	—	—	—	—	
—	—	—	—	—	318	283	276	280·5	273	221·5	228?	252·5	—	140	—	7
—	346	—	345	—	343	—	—	—	—	—	241	—	—	—	—	
362	363	361	361	361	362	324?	328	317	321	252	253	271	274	—	160	7
372	—	372	—	—	—	—	—	—	—	—	—	—	—	—	—	
376	—	374	—	—	368	365	350	358	343	257	258	277	273?	137	154	7
369	368	368	366	354	—	336	323	329	313	—	258	—	273	156	160	
347?	346	343	342	—	336	315	—	312	—	244	247	—	266	—	150	7
348	—	347	—	—	339	314	298	310	295	226	224	—	242	—	—	7
355·5	355·5	354·5	355	347·5	—	320·5	—	313·5	—	236·5?	241·5	256?	257·5	—	150	7
378	379	376	377	367	370	312	309	307	301	245	251	270	271	—	137	7
373	376	371	375	362	358	330	—	328	—	259	259	284	285	148	152	7
381	383	380	380	—	—	340	334	338	333	257	—	280	—	152	—	7
379	—	377	—	—	—	327	325	322	320	267	265	290	286	—	145?	8
—	368	—	366	—	355	330?	—	324?	—	258	—	—	—	—	—	7
335	—	335	—	322	—	308	291	302	283	238	241	260	261	149	157	7
363	363?	362	359?	—	354?	325	313	319	310	254	—	274	—	138?	—	7
380	380	379	379	375	373?	—	—	—	—	—	257	—	—	—	—	
407	411	406	409	405	409	358	—	349	—	—	—	—	—	163	160?	
324	324	324	323	316	—	290	292	288	287	216	218	239	241	132	—	7
353?	—	351?	—	—	—	324?	321?	317?	316?	249	248	—	274	—	—	7
374	375?	373	373?	371	—	332	328	329	323	265	266	[289?]	—	—	156	7
371	377	371	376	367	371	335	326?	330	323?	262	263	285	285	155	158	7
393	—	392	—	390	—	—	333	—	325	—	266	—	293	—	—	
348	352	347	351	346	352	320	313	316	309	—	243	267	268	—	—	
321?	322	320?	321	316	312	304	301	300	298	225	225	245	244	—	—	7
352	353	351	350	342	339	312?	305	310?	304	241	240	261	262	144	150	7
394	394	393	392	386	—	353	349	348	349	260	268	284	289	148	157	7
383	386	380	382	368	368?	356	338	349	333	268	272	289	293	—	—	7
361·5?	362	360?	361·5	—	—	331	316·5	326·5	312	245?	249	266·5?	269	—	128	7
347?	—	343·5?	—	336?	—	307	305	301	299·5	238	238	255·5	256·5	134	143	7
354	360	352	358	338	351	318	323	316	320	242	247	—	265	142	—	7
363	370?	359	368?	362	366	323	314	315	307	246	—	275	273	159	166	7
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	
330	—	329	—	318	—	—	294	—	292	222	226	—	245	—	141	
319?	321	318?	320	320	—	296	—	291	—	217	—	238	—	—	—	7
—	—	—	—	—	—	—	294	—	292	—	—	259	259	130	—	
—	349	—	347	346	—	317	303	313	300	246	—	—	—	131	143	7
319	315	319	314	309	309	291	286	288	282	216	215	238	235	130	133	7
323	322	322	320	—	317	288	281	286	278	215	214	—	231	132	—	7
321	318	319	317	320·5?	316?	287·5	282·5	286	281	213	211·5	—	232·5	131	137	7
330	334·5?	326·5	331?	327?	327	300·5	298	297·5	295	217·5	219	—	238	141	139	7
334	331	333	329	327	323	304	301	302	299	226	227	245	249	—	—	7
320	322	317	319	309	311	—	—	—	—	—	—	—	—	—	—	
347	343	346	342	345	339	—	311	—	308	233	228	256	249	—	151	
347	351	346	350	343	348	311	301	307	295	229	232	254	253?	135	138?	7
332	333	331	332	—	325	301	296	296	292	—	212	—	—	141	142	
—	—	—	—	—	—	—	—	—	—	—	—	—	239	—	—	
337	329	335	328	—	323	304	289	302	286	225	224	—	—	144	—	7
368	370	367	367	—	357	323	315	313	308	235	231	258	251?	136	138	7
289	293	288	291	—	286	269	264	268	263	195	191	218	214	131	132	7
352	353	350	351	—	—	321	316	312	309	244	240	—	258?	—	—	7
376	382	374	379	—	—	318	312	312	304	—	—	—	254	—	—	
—	301?	—	291?	—	—	256	253	255	251	185	186	—	203	129	—	7
—	324	—	322	—	—	—	—	—	—	—	—	—	—	—	—	
312	301	311	300	—	—	264	255	261	253	198	196	223	221	121	122	2
337?	339	333	338	331	336	316	300	315	298	224	221	240	235	—	—	2
357	357	355	353	349	—	315	307?	308	304?	231	227	246	—	—	—	2
361	358	361	357	345	—	—	314	—	308	241	236	261	—	—	—	
—	—	—	—	—	—	298	—	295	—	—	—	—	—	141	143	
326	325	324	324	315	—	—	—	—	—	193	188	216	212	—	—	
359	360	358	359	338	—	—	—	—	—	—	—	—	—	—	—	
345	346	342	344	341	344	313	310	307	304	235	—	—	—	146	151	2

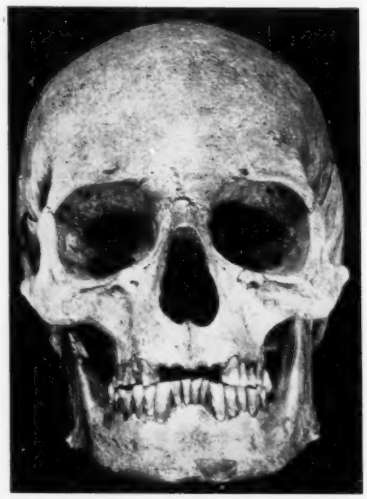
TABLE VIII

f long bones of skeletons from Maiden Castle

Clavicles Maximum		$100 \frac{\text{Radius max.}}{\text{Humerus max.}}$		$100 \frac{\text{Humerus max.}}{\text{Femur obl.}}$		$100 \frac{\text{Tibia obl.}}{\text{Femur obl.}}$		$100 \times \frac{\text{(H. max.} + \text{R. max.)}}{\text{(F. obl.} + \text{T. max. ex spine)}}$		Angle of femur		$100 \frac{\text{Clavicle max.}}{\text{Humerus max.}}$		Recon-strucs stature
R.	L.	R.	L.	R.	L.	R.	L.	R.	L.	R.	L.	R.	L.	
—	—	[76·4]	—	—	—	—	—	—	—	—	—	—	—	1632
—	—	—	78·7	—	—	—	—	—	—	—	—	—	—	1688
140	—	78·3	82·6?	71·2?	—	—	—	—	—	5°·7	—	49·5	—	1558
—	—	—	—	—	—	—	—	—	—	—	—	—	—	1635
—	160	77·8?	77·1	71·8?	72·2	80·0	79·5	70·8?	71·1?	6°·8	7°·6	—	48·8	1652
—	—	—	—	—	—	—	—	—	—	—	—	—	—	1671
137	154	70·4	73·7	80·0	76·1	82·0	—	74·8	—	5°·1	5°·3	37·5	44·0	1713
156	160	—	79·9	76·9	73·4	84·2	83·2	—	71·9	5°·5	5°·4	46·4	49·5	1660
—	150	77·5	—	72·9	—	79·4	78·8	71·8?	—	6°·7	6°·7	—	—	1616
—	—	72·0	75·2	75·3	—	83·2	—	70·6	—	6°·8	—	—	—	1605
—	150	73·8?	—	70·9?	—	78·4?	78·5	69·0?	—	6°·0?	8°·5	—	—	1647
—	137	78·5	81·2	69·5	68·5	83·7	83·6	67·4	67·5	6°·6	3°·8	—	44·3	1646
148	152	78·5	—	—	—	—	—	—	—	—	—	44·8	—	1663
152	—	75·6	—	75·6	73·6	84·4	83·7	71·8	—	6°·6	5°·4	44·7	—	1681
—	145?	81·7	81·5	73·3	71·1	84·5	—	72·0	—	8°·5	6°·6	—	44·6?	1660
—	—	78·2?	—	—	—	—	80·3	—	—	—	7°·6	—	—	1668
149	157	77·3	82·8	72·3	68·3	78·2	—	71·7	—	6°·8	7°·8	49·4	54·0	1596
138?	—	78·2	—	72·4?	70·2?	80·6?	80·5?	71·3?	—	7°·7?	5°·4?	42·5?	—	1653
—	—	—	—	—	—	—	—	—	—	—	—	—	—	1710
163	160?	—	—	—	—	—	83·3	—	—	—	7°·3	45·5	—	1747
132	—	74·5	74·7	73·0	72·8	81·6	80·5	70·2	70·3	8°·1	8°·1	45·5	—	1546
—	—	76·9?	77·3?	—	74·8?	—	—	—	—	—	5°·5	—	—	1626
—	156	79·8	81·1	75·3?	71·9	82·3?	81·8?	72·2?	71·5?	8°·5?	6°·6	—	47·6	1670
155	158	78·2	80·7?	74·4?	71·5?	82·4?	82·5	72·7?	70·7?	3°·6?	6°·6	46·3	48·5?	1666
—	—	—	79·9	—	70·1	83·4?	—	—	—	7°·4?	5°·2	—	—	1702
—	—	—	77·6	76·7	74·5	83·2	83·6	—	72·0	5°·6	5°·6	—	—	1608
—	—	74·0	74·8	74·7	72·4	78·6?	77·2	72·7?	71·3	8°·0	7°·9	—	—	1562
144	150	77·2?	78·7	75·4?	72·8	84·8	83·5	72·2?	70·6	5°·6	4°·0	46·2?	49·2	1593
148	157	73·7	76·8	77·2	76·0	86·0	85·4	72·0	72·3	6°·6	6°·5	41·9	45·0	1711
—	—	75·3	80·5	76·4	73·0	81·5	82·5	73·5	71·8	5°·3	6°·5	—	—	1714
—	128	74·0?	78·7	74·7?	71·0	81·3?	81·1	71·6?	70·0	5°·4?	7°·1	—	40·4	1653
134	143	77·5	78·0	73·4?	—	82·1?	—	71·2?	—	6°·3?	—	43·6	46·9	1595
142	—	76·1	76·5	72·6	73·6	80·4	81·5	70·7	71·3	9°·4	9°·4	44·7	—	1635
159	166	76·2	—	73·1	70·4	81·2	82·5?	70·7	—	5°·4	8°·5	49·2	52·9	1642
—	—	—	—	—	—	—	—	—	—	5°·1?	8°·1?	—	—	1745
—	141	—	76·9	—	72·8	81·0	—	—	—	6°·9	8°·0	—	48·0	1525
—	—	73·3	—	—	—	—	80·8	—	—	—	9°·1	—	—	1502
130	—	—	—	—	70·2	—	—	—	—	—	7°·9	—	—	1541
131	143	77·6	—	—	—	—	—	—	—	—	—	41·3	47·2	1595
130	133	74·2	75·2	77·4	77·3	84·8	84·9	72·9	73·1	8°·3	7°·3	44·7	46·5	1482
132	—	74·7	76·2	74·8	72·4	83·6	82·5	71·0	69·7	7°·1	7°·1	45·8	—	1492
131	137	74·1	74·9	74·3	72·1	82·4	80·9	70·7	69·6	5°·8	6°·5	45·6	48·5	1502
141	139	72·4	73·5	76·2	72·9?	82·8	80·9?	71·5	69·5?	9°·9	9°·3?	46·9	46·6	1522
—	—	74·3	75·4	73·6	72·0	80·6	78·7	71·0	70·5	4°·0	6°·8	—	—	1534
—	—	—	—	—	—	79·8	—	—	—	8°·1	—	—	—	1502
—	151	—	73·3	—	72·8	80·8	80·1	—	70·0	7°·8	6°·8	—	48·6	1579
135	138?	73·6	77·1	72·7	69·4?	80·8	80·6?	69·6	67·9?	7°·8	10°·9?	43·4	45·8?	1562
141	142	—	71·6	74·9	—	82·3	—	—	—	8°·1	—	46·8	48·0	1532
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
144	—	74·0	77·5	—	71·9	—	81·6	—	70·2	—	7°·0	47·4	—	1532
136	138	72·8	73·3	71·8	69·5	81·6	81·0	68·2	66·3	8°·5	7°·6	42·1	43·8	1612
131	132	72·5	72·3	72·7	70·6?	77·8	77·8?	70·4	68·2?	7°·3	9°·3?	48·7	50·0	1432
—	—	76·0	75·9	70·7	69·8	77·1	77·5	70·1	69·0	6°·6	6°·6	—	—	1592
—	—	—	—	72·9	70·9	85·8	86·1	—	—	11°·5	11°·5	—	—	1602
129	—	72·3	73·5	70·5?	69·3	—	79·7?	—	65·9?	6°·0?	9°·4	50·4	—	1432
—	—	—	—	—	—	—	81·7	—	—	9°·9	9°·1	—	—	1512
121	122	75·0	76·9	72·3	69·1	85·2	81·3	68·2	67·3	7°·3	7°·3	45·8	47·8	1452
—	—	70·9	73·7	75·4	70·6?	79·5	79·5?	71·4?	68·2?	7°·9	6°·8?	—	—	1552
—	—	73·3	73·9?	71·8	70·1?	80·9	80·6	68·6	67·2?	6°·7	9°·4	—	—	1592
—	—	—	75·2	—	71·5	81·9	81·3	—	69·0	6°·7	5°·5	—	—	1592
141	143	—	—	—	—	—	—	—	—	—	—	47·3	—	1532
—	—	—	—	—	—	78·3	78·6	—	—	6°·9	8°·9	—	—	1502
—	—	—	—	—	—	—	—	—	—	—	—	—	—	1632
146	151	75·1	—	71·8	71·4	78·4	79·3	70·2	—	5°·5	9°·5	46·6	48·7	1562

ine)	Angle of femur		100 Clavicle max. / Humerus max.		Reconstructed stature	No.	Remarks
	R.	L.	R.	L.			
	—	—	—	—	1632	Q 1	Most of the long bones are mutilated
	—	—	—	—	1688	Pit 3	—
	5°·7	—	49·5	—	1558	O 3	—
	—	—	—	—	1635	T 3	—
	6°·8	7°·6	—	48·8	1652	T 4	—
	—	—	—	—	1671	T 5	Severe injury to R. elbow joint, and healed fractures of L. radius (near wrist) and L. fibula (near ankle), the latter bone being fused to the tibia
	5°·1	5°·3	37·5	44·0	1713	T 10	R. clavicle markedly shorter and thicker than L.
	5°·5	5°·4	46·4	49·5	1660	T 11	—
	6°·7	6°·7	—	—	1616	T 20	—
	6°·8	—	—	—	1605	T 25	—
	6°·0?	8°·5	—	—	1647	N 1	—
	6°·6	3°·8	—	44·3	1646	P 2	—
	—	—	44·8	—	1663	P 6	—
	6°·6	5°·4	44·7	—	1681	P 7	—
	8°·5	6°·6	—	44·6?	1660	P 7A	Iron arrow-head embedded in twelfth thoracic vertebra
	—	7°·6	—	—	1668	P 8	—
	6°·8	7°·8	49·4	54·0	1596	P 9	—
	7°·7?	5°·4?	42·5?	—	1653	P 11	—
	—	—	—	—	1710	P 12	—
	—	7°·3	45·5	—	1747	P 18	—
	8°·1	8°·1	45·5	—	1546	P 19A	—
	—	5°·5	—	—	1626	P 22	—
	8°·5?	6°·6	—	47·6	1670	P 23	R. ulna fractured
	3°·6?	6°·6	46·3	48·5?	1666	P 24	—
	7°·4?	5°·2	—	—	1702	P 25	Osseous deposits on muscular ridges of L. femur and L. fibula
	5°·6	5°·6	—	—	1608	P 27	—
	8°·0	7°·9	—	—	1569	P 28	—
	5°·6	4°·0	46·2?	49·2	1599	P 29	—
	6°·6	6°·5	41·9	45·0	1711	P 30	—
	5°·3	6°·5	—	—	1714	P 34	—
	5°·4?	7°·1	—	40·4	1653	O 1	—
	6°·3?	—	43·6	46·9	1595	O 2	—
	9°·4	9°·4	44·7	—	1633	QR V	—
	5°·4	8°·5	49·2	52·9	1643	T 16	—
	5°·1?	8°·1?	—	—	1749	Q	—
	6°·9	8°·0	—	48·0	1523	T 1	—
	—	9°·1	—	—	1505	T 13	—
	—	7°·9	—	—	1545	T 14	—
	—	—	41·3	47·2	1598	T 18	—
	8°·3	7°·3	44·7	46·5	1484	T 24	—
	7°·1	7°·1	45·8	—	1492	Q 1	—
	5°·8	6°·5	45·6	48·5	1502	O 4	—
	9°·9	9°·3?	46·9	46·6	1521	N 2	—
	4°·0	6°·8	—	—	1534	T 6	—
	8°·1	—	—	—	1504	T 12	L. humerus markedly deformed, probably by inflammatory condition of soft tissues
	7°·8	6°·8	—	48·6	1570	T 21	—
	7°·8	10°·9?	43·4	45·8?	1568	T 22	—
	8°·1	—	46·8	48·0	1531	T 26	—
	—	—	—	—	—	T 27	—
	—	7°·0	47·4	—	1531	P 5	—
	8°·5	7°·6	42·1	43·8	1618	P 14	—
	7°·3	9°·3?	48·7	50·0	1436	P 16	—
	6°·6	6°·6	—	—	1593	P 19	—
	11°·5	11°·5	—	—	1606	P 26	—
	6°·0?	9°·4	50·4	—	1438	P 31	—
	9°·9	9°·1	—	—	1511	P 33	—
	7°·3	7°·3	45·8	47·8	1451	P 36	Healed fracture of L. fibula
	7°·9	6°·8?	—	—	1558	P 37	—
	6°·7	9°·4	—	—	1590	T 28	—
	6°·7	5°·5	—	—	1593	QR II	—
	—	—	47·3	—	1535	QR III	—
	6°·9	8°·9	—	—	1504	QR IV	—
	—	—	—	—	1634	T 17	—
	5°·5	9°·5	46·6	48·7	1568	P 20	Epiphyses of ulnae not attached

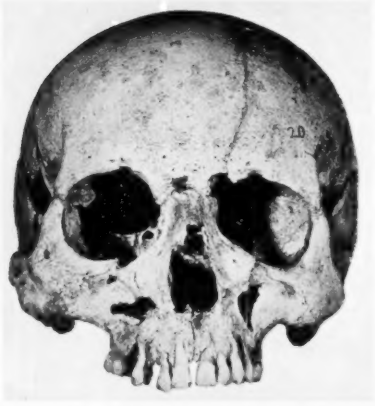
A



B

C

A typical male skull (P 30) from the Belgic War Cemetery.



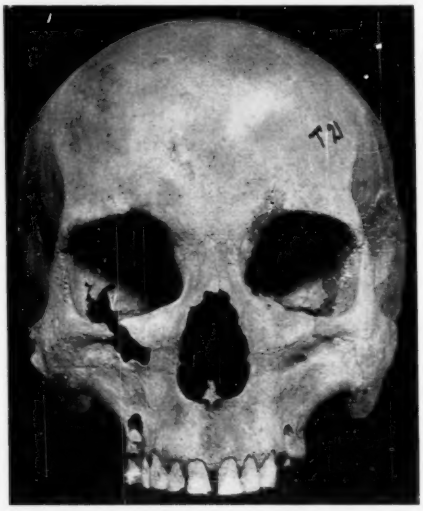
A. P 20, female. Injury to right malar bone.

B. P 7 A, male. Trace of suture on right parietal bone.

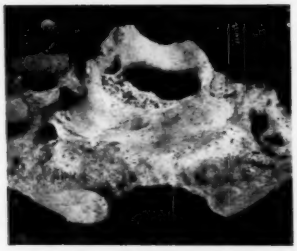
C. P 36, female. Exceptional type of facial skeleton and high cephalic index (87·0).

D. P 36, female. Profile view of the same skull (C).

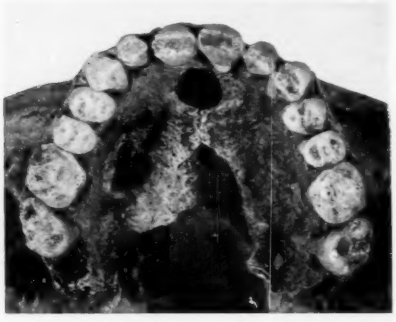
Exceptional skulls from the Belgic War Cemetery.



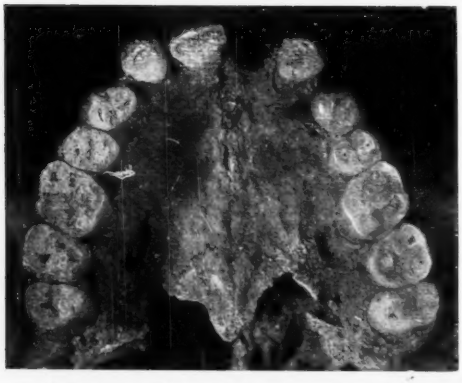
A. T 21, female. Cyst cavities at roots of second premolars.



B. O 3, male. Broken basi-occipital showing the extension into it of sphenoidal air sinuses.



C. P 7, male. Large anterior palatine foramen.



D. T 28, female. Absence of canines and first premolars rotated.

E. P 14, female. Left third molar impacted and mandibular torus.

Skulls from Maiden Castle with dental and other anomalies.

HOMOGENEITY OF RESULTS IN TESTING SAMPLES FROM POISSON SERIES

WITH AN APPLICATION TO TESTING CLOVER SEED FOR DODDER

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1. Introductory

In our previous paper, "Statistical principles of routine work in testing clover seed for dodder" (Przyborowski & Wilenski, 1935), we justified the assumption that the number of dodder seeds in samples of clover does follow the Poisson Law:

$$p(x) = e^{-m}\frac{m^x}{x!} \quad (x = 0, 1, 2, \ldots). \tag{1}$$

Thus we constructed rules which should be followed in sampling problems where that Law holds good. We analysed particularly the application of the rules to the routine work in testing clover seed for dodder.

The purpose of this paper is to present some theoretical considerations relating to the problem of homogeneity of results in testing samples drawn from Poisson series which we have found arising in the course of our work on testing clover seed.

Let m_1 and m_2 denote the parameters of the Poisson distributions of the variables x_1 and x_2; the problem to be considered consists in testing the hypothesis $H_0(m_1 = m_2 = m)$, that the values x_1 and x_2 were obtained in sampling from Poisson series, the parameters of which have a common but unspecified value, say m.

The method of testing statistical hypotheses as developed by Neyman & Pearson (1933) consists in selecting a rule of rejecting the hypothesis in question, whenever the sample point E (the co-ordinates of which in the n-dimensional sample space W are the data of observation $x_1, x_2, \ldots, x_n$) lies within the so-called critical region, say w, of the sample space W. The probability, $P\{E \epsilon w \mid H_0\} = \alpha$, of rejecting the hypothesis tested, when it is true, is called the size of the corresponding critical region w on which the test is based. The errors thus committed when rejecting the true hypothesis H_0, are called the errors of the first kind. The probability $P\{E \epsilon w \mid H'\}$ of rejecting the hypothesis H_0 when an alternative H' is true has been termed the power of the test with regard to H'. $P\{E \epsilon w \mid H'\}$ considered as a function of H', where H' is any hypothesis belonging to the class Ω of alternatives to H_0, is called the power function of the test. The error committed when we fail to reject H_0 although

an alternative, H', is true, has been termed an error of the second kind. The risk of committing this error is $1-P\{E\epsilon w \mid H'\}$.

The general guiding principle to be followed in selecting a test of H_0 is to choose out of critical regions of size α, a region which makes the power of the test with regard to the admissible alternative hypotheses of Ω as large as possible. In certain cases there will be a single region, say w_0, which maximizes the function $P\{E\epsilon w \mid H'\}$ for every admissible alternative. The test based on this region, if it exists, is called the uniformly most powerful test of H_0 with regard to the set Ω. In very many problems, however, and the present is one of them, no such region exists. We shall now consider in our problem what form of test is possible, and what are some of its properties. In doing this, it should be pointed out that because the random variables x_1 and x_2 can only assume integral values, the theory of unbiased tests as developed by Neyman & Pearson is not strictly applicable, although the test we shall reach is analogous to the unbiased test of type B (Neyman, 1935).

2. Derivation of test

Suppose, in the first place, that m_1 does not necessarily equal m_2 and that we write

$$n = x_1 + x_2, \qquad \text{.....(2)}$$

$$\mu = m_1 + m_2, \qquad \text{.....(3)}$$

$$\rho = m_1/(m_1+m_2). \qquad \text{.....(4)}$$

Then the joint elementary probability law of the observations x_1, x_2 takes the form

$$p(x_1, x_2 \mid m_1, m_2) = \frac{m_1^{x_1} m_2^{x_2} e^{-(m_1+m_2)}}{x_1!\, x_2!}, \qquad \text{.....(5)}$$

or

$$p(x_1, x_2 \mid \rho, \mu) = \frac{\mu^n e^{-\mu}}{n!} \times \frac{n!}{x_1!(n-x_1)!} \rho^{x_1}(1-\rho)^{n-x_1}. \qquad \text{.....(6)}$$

The sample space is a two-dimensional one as shown in Fig. 1, and since the values of x_1 and x_2 are necessarily integral, the possible sample points E are distributed discretely as indicated. Along each diagonal line, $n = x_1 + x_2 = $ constant. The form of (6) shows that the probability law is the product of two terms:

(1) The chance $$p(n \mid \mu) = \frac{\mu^n e^{-\mu}}{n!}, \qquad \text{.....(7)}$$

that E falls on the line (2), which depends only on μ;

(2) The chance that for a given n, the observed partition into x_1 and $x_2 = n - x_1$ occurs, or

$$p(x_1 \mid n, \rho) = \frac{n!}{x_1!(n-x_1)!} \rho^{x_1}(1-\rho)^{n-x_1}. \qquad \text{.....(8)}$$

This is a term of the binomial expansion $\{(1-\rho)+\rho\}^n$, depending on ρ, but not on μ.

The hypothesis to be tested, H_0, is that $m_1 = m_2$ or that $\rho = \frac{1}{2}$; we shall suppose that the admissible alternatives to H_0 are both that $m_1 > m_2$ and $m_1 < m_2$ or that $\rho < \frac{1}{2}$ and $\rho > \frac{1}{2}$. Since H_0 does not specify the value of μ (which may have any value between 0 and ∞), it is a composite hypothesis in the

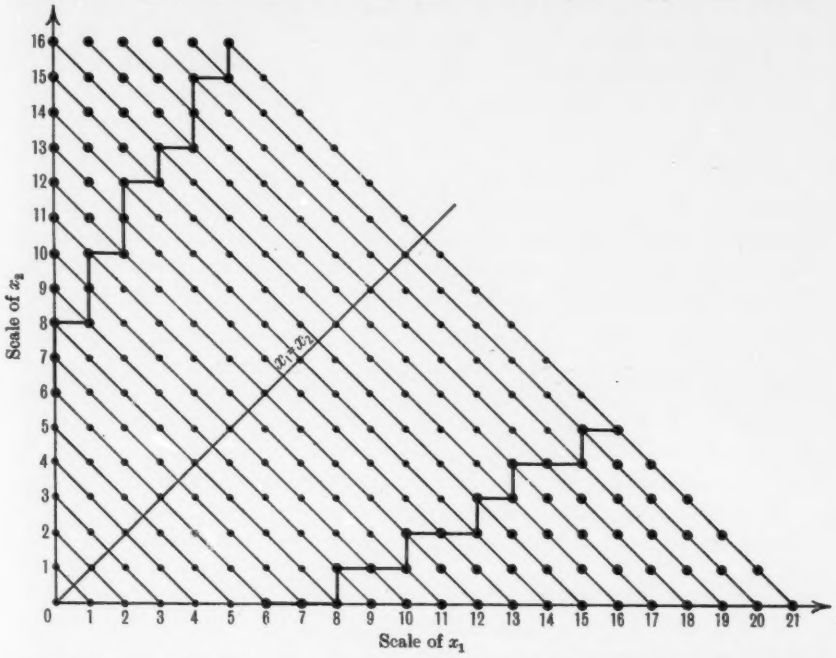

Fig. 1. Critical region for case α=0·05. It consists of the two areas containing the larger dots.

sense of Neyman & Pearson. To test it, we should like to be able to use a critical region, w, such that

$$P\{E\epsilon w \mid \rho = \tfrac{1}{2}, \mu\} = \alpha, \qquad \text{.....(9)}$$

whatever be μ. No such region can be found, owing to the discontinuous character of the distribution of the variables, but it is possible to go a long way in determining a satisfactory region on the following lines.

It will be noted that if $\rho = \frac{1}{2}$, then the distribution of x_1 on each line, n = constant, is that of the symmetrical binomial $(\frac{1}{2} + \frac{1}{2})^n$. If ρ is either $> \frac{1}{2}$ or $< \frac{1}{2}$, we shall have a skew binomial with probability density concentrating towards the lower or upper end of the line; in order to make the test as efficient as possible in detecting departures from $\rho = \frac{1}{2}$ in either direction, the critical or rejection region should therefore include the two tails of each of the separate binomials $(\frac{1}{2} + \frac{1}{2})^n$. It may be defined as follows.

Build up the critical region w of "pieces", $w(n, \alpha)$, of the lines

$$n = x_1 + x_2 = \text{constant} \quad (n = 0, 1, 2, \ldots), \qquad \ldots\ldots(10)$$

where $w(n, \alpha)$ includes all sample points on (10) for which

$$x_1 \leqslant k(n, \alpha) \quad \text{and} \quad x_1 \geqslant n - k(n, \alpha), \qquad \ldots\ldots(11)$$

$k(n, \alpha)$ being a positive integer determined so that:

(*a*) the sum of the terms of the binomial $(\frac{1}{2} + \frac{1}{2})^n$ corresponding to

$$0, 1, \ldots, \quad k(n, \alpha) \text{ is } \leqslant \tfrac{1}{2}\alpha;$$

(*b*) the sum of the terms corresponding to

$$0, 1, \ldots, \quad k(n, \alpha), \quad k(n, \alpha) + 1 \text{ is } > \tfrac{1}{2}\alpha.$$

The pieces $w(n, \alpha)$ and part of the complete region w for the case $\alpha = 0{\cdot}05$ are indicated in Fig. 1. In view of the form of the probability law (6) it will be seen that the region w is such that

$$P\{E \epsilon w \mid \rho = \tfrac{1}{2}, \mu\} \leqslant \alpha, \qquad \ldots\ldots(12)$$

whatever be μ. Thus if we reject the hypothesis H_0 which assumes $m_1 = m_2$, whenever the sample point is included in w, i.e. falls at any of the points indicated in Fig. 1 by the larger spots, we know that the risk of the first kind of error is at most equal to α. Further, while it cannot be claimed that the test is completely unbiased or is the uniformly most powerful unbiased test,* it seems likely to be as efficient as any other alternative test in detecting departures in ρ from $\frac{1}{2}$.

In Table I we have given the boundary values $k(n, \alpha)$ of the critical regions associated with four values of α, viz. 0·20, 0·10, 0·05, 0·01, and for $n = x_1 + x_2$ varying from 0 to 80. Having decided on the appropriate value for α, the rule of the test is: reject the hypothesis that $m_1 = m_2$ if

$$x_1 \leqslant k(n, \alpha) \quad \text{or} \quad x_1 \geqslant n - k(n, \alpha).$$

It will be seen that for small values of n no test associated with the α required may be possible.

The power function of the test, which depends on both μ and ρ, may be calculated from the expression

$$P\{E \epsilon w \mid \rho, \mu\} = \sum_{n=0}^{\infty} \left\{ \frac{\mu^n e^{-\mu}}{n!} \sum_{w(n,\alpha)} \frac{n!}{x_1!(n-x_1)!} \rho^{x_1} (1-\rho)^{n-x_1} \right\}, \qquad \ldots\ldots(13)$$

where $\sum_{w(n,\alpha)}$ is the sum of the terms of the binomial expansion

$$\{(1-\rho) + \rho\}^n,$$

satisfying the conditions (11).

Values of the power for the two cases $\alpha = 0{\cdot}10$ and $\alpha = 0{\cdot}05$, and for a number of values of μ and ρ are given in Table II *a* and *b*. The computation involved (*a*) first finding for a given n the sums of the binomial terms in the second summation of (13) with the help of the *Tables of the Incomplete Beta Function*

* These terms have been applied to problems of continuous variables.

(1934), and (*b*) then multiplying this by the appropriate term of the Poisson series, and (*c*) finally summing for $n = 0, 1, 2, \ldots$.

It will be noted that if the admissible alternatives to $m_1 = m_2$ are only that $m_1 > m_2$, then the appropriate critical region will be defined by $x_1 \geqslant n - k(n, \alpha)$, and the risk of the first kind of error will now be $\leqslant \frac{1}{2}\alpha$. Similarly, if the alternatives are only $m_1 < m_2$, the region will be defined by $x_1 \leqslant k(n, \alpha)$. These regions correspond to the lower and upper marked areas, respectively, of Fig. 1.

We have also given in Tables II in certain cases the actual value of $P\{E\epsilon w \mid \rho = \frac{1}{2}, \mu\}$, that is to say, the chance of rejecting the hypothesis that $m_1 = m_2$ when it is true. These chances are, as expected, considerably less than the corresponding values of α, but approach α as μ increases and the discontinuity in the distributions of x_1 and x_2 becomes of less importance. In practice we shall not, of course, know exactly what μ is, but as we shall probably have a rough idea of the size of the Poisson m in the particular problems investigated, the figures in these columns of the Tables will probably be helpful in determining the value to select for α, the upper limit of the significance level. Table III contains true significance levels for the other two limiting values, $\alpha = 0{\cdot}20$ and $\alpha = 0{\cdot}01$, for which the rejection levels $k(n, \alpha)$ have been given in Table I.

From the data of Tables II it was possible to construct the charts given in Figs. 2 and 3 (pp. 322, 323 below), showing in terms of the Poisson parameters m_1 and m_2 (rather than μ and ρ) some of the contours of equal power.

3. Discussion of tables and charts with some illustrative examples

A point which is clearly brought out by the present investigation is that it is impossible to detect differences between the means of two Poisson series m_1 and m_2, when both these quantities are small, even if one is several times larger than the other. Thus even were $m_1 = 0$ and $m_2 = 5$, the chance of detecting the difference from the samples would be only 0·384 using $\alpha = 0{\cdot}05$, and 0·560 using $\alpha = 0{\cdot}10$.* As an illustration of the kind of differences that may be detected, Fig. 2 may be used as follows. It might with reason be regarded as undesirable to plan an experiment in which the chance was less than 0·5 of detecting from two random samples that differences between m_1 and m_2 of practical importance existed. If we pass along the 50 % power contour in the diagram, we pass through points (m_1, m_2) of which the following are typical:

$$(0{\cdot}0,\ 4{\cdot}7),\quad (2{\cdot}0,\ 8{\cdot}1),\quad (6{\cdot}0,\ 14{\cdot}3),\quad (10{\cdot}0,\ 19{\cdot}8),\quad (18{\cdot}0,\ 30{\cdot}5).$$

If it is important to be able to detect differences of this magnitude, then larger samples must be taken in order to increase the expectations. If the samples

* The true values of the significance level in these cases would be 0·09 and 0·024 respectively; if the test were used with $\alpha = 0{\cdot}20$ (and a significance level in this case of 0·065) the chance of detection would be somewhat greater.

can, for instance, be increased c-fold, the expectations will be increased to cm_1, cm_2, a point lying on the same straight line passing through the origin as m_1, m_2, but placed c times further out. Since such radial lines will cut the contours of equal power at rather acute angles, considerable multiplication of sample sizes may often be needed to ensure detection of differences.

Regarding a chance of detection of 19 to 1 as satisfactory, we note that the 95% contour in Fig. 2 passes through points (m_1, m_2) of which the following are typical:

(0·0, 9·1), (2·0, 15·5), (6·0, 24·0), (10·0, 31·5).

Differences of this kind, if they exist, can therefore be almost certainly detected using in the test the critical values associated with $\alpha = 0{\cdot}10$. Rather smaller differences could be detected with the same high probability if α were taken as 0·20.

In practice, of course, expected values m_1 and m_2 are not known in advance, but we believe that reasonings of this kind based on the tables or charts may be useful as a preliminary step in planning the extent of sampling required either in experimental work or routine analysis.

Example. In the testing of clover seed for dodder it is customary to withdraw a 100 gr. sample from a sack with a long trier or probe. Clearly the number of dodder seeds found will vary from sample to sample, and from one sack to another in the same consignment. This may be due to the following causes:

(1) The material from which the samples were drawn was not thoroughly mixed.

(2) The two samples were drawn from different seed stocks.

(3) There are laboratory errors in analysis.

(4) The existence of random sampling fluctuations.

In applying statistical analysis in the comparison of two counts, the hypothesis to be tested assumes that the discrepancy to be tested is merely due to chance, and that the frequency of dodder seeds, x, will vary from one sample to another in accordance with the Poisson law. If a significant difference is found, it may of course be due to one of the first three sources of error.

Suppose that a 100 gr. sample is drawn from a sealed sack of clover and sent to a seed testing station, and that no dodder seeds are found in it, i.e. $x_1 = 0$. The buyer, before opening the sack, takes another 100 gr. sample, and on having it analysed learns that three seeds have been found, i.e. $x_2 = 3$. Would he have any justification for criticizing the first testing on the grounds that the difference between 0 and 3 could not be due to chance? On examining Table I, it is seen that for $n = x_1 + x_2 = 3$, even using the least stringent of the four tests, the difference cannot be regarded as significant.*

Suppose that on another occasion it was found that $x_1 = 2$ and $x_2 = 6$.

* It will be noted from Table III that with μ equal to 5 or less, the true significance level for the first test with $\alpha = 0{\cdot}20$ is at about 0·06.

Entering Table I with $n = 8$, we see that for none of the levels of test does $x_1 = 2$ fall in the critical region. If the purchaser still considered that to find three times as many dodder seeds in one sample as in the other could not be due to chance, we might illustrate the danger of drawing conclusions in this way by a use of one of the charts. If one of the *expected* numbers were three times the other, so that $m_2 = 3m_1$, and therefore $\rho = 0{\cdot}25$, we may ask how large $\mu = m_1 + m_2$ must be before there is as much as an even chance of detecting the difference. Taking Fig. 3, it is found that the line $m_2 = 3m_1$ cuts the 50 % power contour at about the point $m_2 = 13{\cdot}8$, $m_1 = 4{\cdot}6$. Thus samples of clover at least twice as large as those taken would be needed to provide only a 50 : 50 chance of detecting a real difference in dodder content of the order suspected by the purchaser. In this case the true significance level of the test used is seen from the number on the diagonal line of the chart to be about 0·03.

To be almost certain of detecting a difference with $m_2 = 3m_1$, for example to make the odds 19 to 1, we notice that a continuation of the same line cuts the 95 % power contour where $m_2 = 38{\cdot}4$, $m_1 = 12{\cdot}8$. For this samples of about 600 gr. would be needed.

4. SUMMARY

The problem considered is that in which x_1 and x_2 are two independent random variables distributed in accordance with the Poisson law of equation (1), and it is desired to test the hypothesis that the expectations m_1 and m_2 are the same. We have shown how a test may be derived which is independent of the value of the unknown common hypothetical expectation but which, owing to the discontinuous nature of the probability distributions, will only provide an upper limit to the significance level, i.e. to the chance of rejecting the hypothesis tested when it is true. The manner of approach of the significance level to its upper limit has, however, been investigated numerically.

A table (Table I) has been provided, containing critical values $k(n, \alpha)$ required in carrying out the test. The power function of the test has also been determined, and tables (Table II *a*, *b*) and charts (Figs. 2, 3) given which make it possible to determine the chance of detecting differences in the expectations m_1 and m_2 of specified magnitudes. Finally, a discussion of some uses of the test has been added.

In conclusion we should like to express our thanks to Prof. E. S. Pearson for his helpful suggestions and criticism made during the course of our investigation.*

* [Certain modifications and additions to the paper have been made since it was received for publication at the beginning of July 1939. As circumstances have unfortunately made communication with the authors impossible, I must accept responsibility for these alterations, which have been mainly concerned with the extension of Tables I and II to higher values of n and μ. E. S. P.]

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TABLE I

Boundary values, $k(n, \alpha)$, for each n

$n = x_1 + x_2$	Upper limit of significance level, α				$n = x_1 + x_2$	Upper limit of significance level, α			
	0·20	0·10	0·05	0·01		0·20	0·10	0·05	0·01
1					41	15	14	13	11
2					42	16	15	14	12
3					43	16	15	14	12
4	0				44	17	16	15	13
5	0	0			45	17	16	15	13
6	0	0	0		46	18	16	15	13
7	1	0	0		47	18	17	16	14
8	1	1	0	0	48	19	17	16	14
9	2	1	1	0	49	19	18	17	15
10	2	1	1	0	50	19	18	17	15
11	2	2	1	0	51	20	19	18	15
12	3	2	2	1	52	20	19	18	16
13	3	3	2	1	53	21	20	18	16
14	4	3	2	1	54	21	20	19	17
15	4	3	3	2	55	22	20	19	17
16	4	4	3	2	56	22	21	20	17
17	5	4	4	2	57	23	21	20	18
18	5	5	4	3	58	23	22	21	18
19	6	5	4	3	59	24	22	21	19
20	6	5	5	3	60	24	23	21	19
21	7	6	5	4	61	24	23	22	20
22	7	6	5	4	62	25	24	22	20
23	7	7	6	4	63	25	24	23	20
24	8	7	6	5	64	26	24	23	21
25	8	7	7	5	65	26	25	24	21
26	9	8	7	6	66	27	25	24	22
27	9	8	7	6	67	27	26	25	22
28	10	9	8	6	68	28	26	25	22
29	10	9	8	7	69	28	27	25	23
30	10	10	9	7	70	29	27	26	23
31	11	10	9	7	71	29	28	26	24
32	11	10	9	8	72	30	28	27	24
33	12	11	10	8	73	30	28	27	25
34	12	11	10	9	74	30	29	28	25
35	13	12	11	9	75	31	29	28	25
36	13	12	11	9	76	31	30	28	26
37	14	13	12	10	77	32	30	29	26
38	14	13	12	10	78	32	31	29	27
39	15	13	12	11	79	33	31	30	27
40	15	14	13	11	80	33	32	30	28

Rule of test: reject hypothesis that $m_1 = m_2$ if $x_1 \leqslant k(n, \alpha)$ or $x_1 \geqslant n - k(n, \alpha)$.

See note on p. 323 regarding extension of limits for $n > 80$.

TABLE II *a*

Power of test; upper limit of significance level, $\alpha = 0{\cdot}10$

$\mu = m_1 + m_2$	$\rho = 0{\cdot}5$ (hypothesis true)	$\rho = 0{\cdot}4$	$\rho = 0{\cdot}3$	$\rho = 0{\cdot}2$	$\rho = 0{\cdot}1$	$\rho = 0{\cdot}0$ ($m_1 = 0$, $m_2 = \mu$)
2	—	0·004	0·008	0·016	0·030	0·053
3	—	0·013	0·028	0·057	0·106	0·185
4	—	0·026	0·056	0·116	0·217	0·371
5	0·024	0·039	0·087	0·182	0·338	0·560
6	—	0·050	0·117	0·248	0·453	0·715
7	—	0·061	0·147	0·311	0·553	0·827
8	—	0·071	0·176	0·371	0·638	0·900
9	—	0·081	0·205	0·424	0·709	0·945
10	0·047	0·093	0·235	0·482	0·768	0·971
11	—	0·100	0·264	0·533	0·817	0·985
12	—	0·110	0·292	0·580	0·856	0·992
13	—	0·119	0·320	0·624	0·888	0·996
14	—	0·128	0·347	0·664	0·913	0·999
15	0·059	0·136	0·373	0·700	0·933	0·999
20	0·065	0·172	0·482	0·831	0·983	1·000
25	0·068	0·205	0·579	0·906	0·996	—
30	0·071	0·239	0·661	0·949	0·999	—
35	0·073	0·272	0·732	0·973	1·000	—
40	0·074	0·301	0·784	0·986	—	—
45	0·074	0·330	0·829	0·993	—	—
50	0·075	0·359	0·865	0·996	—	—

TABLE II *b*

Power of test; upper limit of significance level, $\alpha = 0{\cdot}05$

$\mu = m_1 + m_2$	$\rho = 0{\cdot}5$ (hypothesis true)	$\rho = 0{\cdot}4$	$\rho = 0{\cdot}3$	$\rho = 0{\cdot}2$	$\rho = 0{\cdot}1$	$\rho = 0{\cdot}0$ ($m_1 = 0$, $m_2 = \mu$)
2	—	0·001	0·002	0·004	0·009	0·017
3	—	0·004	0·009	0·021	0·044	0·084
4	—	0·009	0·023	0·054	0·113	0·215
5	0·009	0·016	0·042	0·099	0·208	0·384
6	—	0·023	0·063	0·152	0·313	0·554
7	—	0·031	0·085	0·206	0·416	0·699
8	—	0·037	0·107	0·260	0·510	0·809
9	—	0·043	0·129	0·312	0·593	0·884
10	0·022	0·049	0·150	0·361	0·664	0·933
11	—	0·054	0·171	0·408	0·724	0·962
12	—	0·060	0·192	0·453	0·774	0·980
13	—	0·065	0·213	0·496	0·817	0·989
14	—	0·070	0·235	0·537	0·852	0·994
15	0·027	0·076	0·256	0·576	0·881	0·997
20	0·029	0·099	0·353	0·731	0·962	1·000
25	0·030	0·120	0·440	0·833	0·989	—
30	0·031	0·143	0·524	0·900	0·997	—
35	0·033	0·170	0·604	0·943	0·999	—
40	0·034	0·198	0·675	0·969	1·000	—
45	0·036	0·224	0·735	0·983	—	—
50	0·037	0·250	0·785	0·991	—	—

TABLE III

Upper limit of significance level $\alpha = 0{\cdot}20$		Upper limit of significance level $\alpha = 0{\cdot}01$	
μ	True level	μ	True level
5	0·065	5	0·0007
10	0·109	10	0·0031
15	0·123	15	0·0042
20	0·132	20	0·0048
25	0·137	25	0·0054
30	0·141	30	0·0058
35	0·148	35	0·0062
40	0·152	40	0·0065
45	0·154	45	0·0067
50	0·155	50	0·0068

N.B. If hypothesis tested is true, $\mu = 2m_1 = 2m_2$.

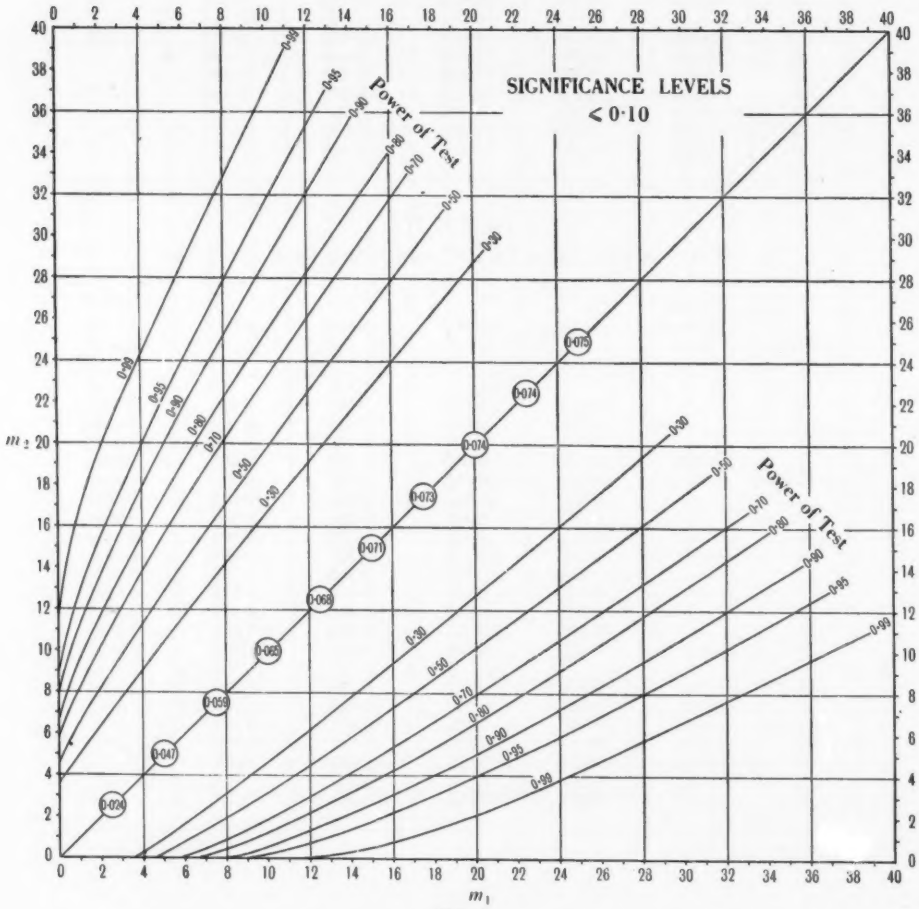

Fig. 2

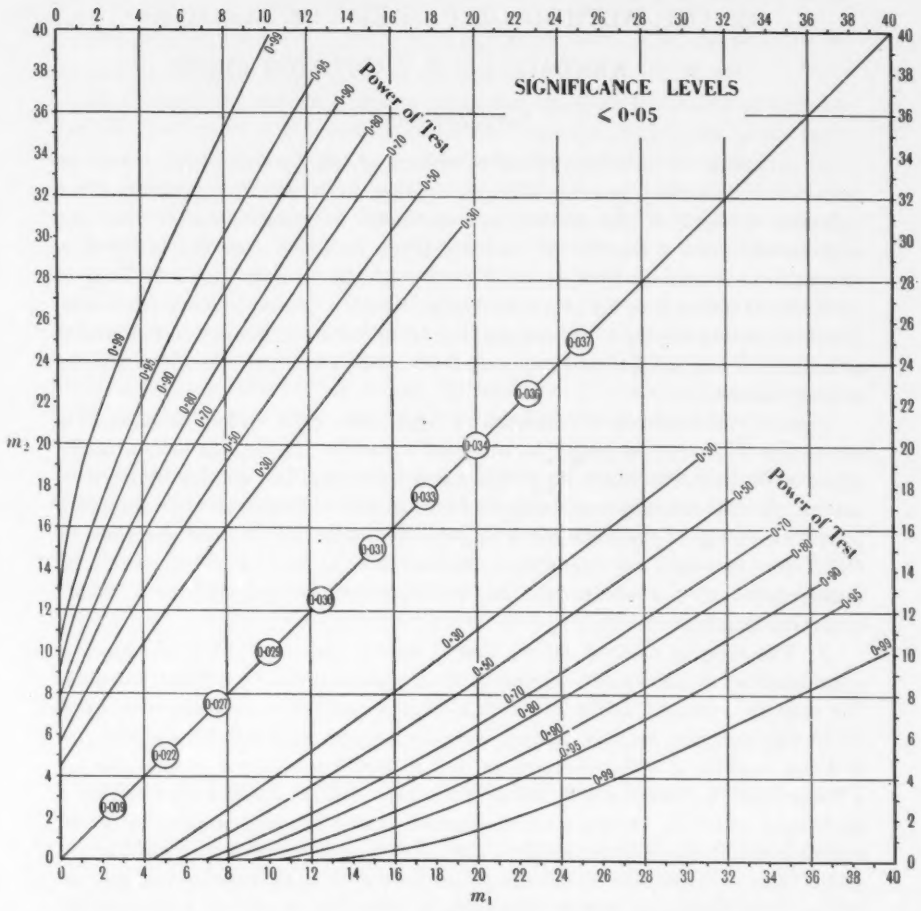

Fig. 3

Note regarding Table I. When x_1 and x_2 are large, if the hypothesis tested is true then $(x_1-x_2)/\sqrt{(x_1+x_2)}$ may be regarded as a unit normal deviate or, from another point of view, x_1 should be normally distributed about $\frac{1}{2}n$ with standard deviation $\frac{1}{2}\sqrt{n}$. On this basis, in the case of $n=80$, the values of $k(n, \alpha)$ for $\alpha=0{\cdot}20$, $0{\cdot}10$, $0{\cdot}05$ and $0{\cdot}01$ are found to be 34·3, 32·6, 31·2 and 28·5, respectively. Applying a correction for discontinuity we have 33, 32, 30 and 28 for the integral values of $k(n, \alpha)$ associated with upper limits of significance level equal to the four values of α. These are the values for $k(n, \alpha)$ given in the last row of Table I. It is clear that beyond $n=80$ the normal approximation is valid.

ON THE METHOD OF PAIRED COMPARISONS

By M. G. KENDALL and B. BABINGTON SMITH

Introduction

1. Suppose we have a number of objects A, B, C, etc. which are to be considered according to the different degrees in which they exhibit some common quality. If the quality is measurable in some objective way the objects will yield a number of variate values, in which case the problem is amenable to treatment by well-understood methods. It may, however, happen either for theoretical or for practical reasons that the quality is not measurable. We then have to rely for a discussion of the variation of the quality on judgments of a more or less subjective kind carried out after a comparison of the objects among themselves.

One of the methods of comparison which has been widely used in this connexion is that of ranking. An observer examines the objects and arranges them in the order in which he judges them to possess the quality under consideration. This arrangement is called a ranking, and when two or more observers provide rankings of the same set of objects there arise the familiar questions of the type: is there any significant resemblance between the judgments of observers? or, do the data furnish any evidence that the objects have a "real" objective ranking?

2. The ranking method suffers from a serious drawback when the quality considered is not known with certainty to be representable by a linear variable. We may, for instance, ask an observer to rank a number of individuals in order of intelligence, and he may comply with the request in the full belief that he is doing something within his powers; but if intelligence is not measurable on a linear scale this ranking may fail to give a real picture either of the observer's preference or of the variation of intelligence among the individuals. It is not impossible that the observer should judge A more intelligent than B, B than C, and C than A, if the individuals are presented for his consideration one pair at a time. The likelihood of this happening is obviously increased when we are dealing with tastes in music, eatables or film stars; and in practice the event is not uncommon. Such "inconsistent" preferences can never appear in ranking, for if A is preferred to B and B to C, then A must automatically be shown as preferred to C. The use of ranking thus destroys what may be valuable information about preferences.

3. In this paper we consider a more general method of investigating preferences. With n objects, we shall suppose that each of the $\binom{n}{2}$ possible pairs is presented to an observer and his preference of one member of the pair noted.

We assume that a choice between two objects can always be made.* With m observers the data then comprise $m\binom{n}{2}$ preferences. The questions to be discussed include:

(*a*) Is there any evidence that a particular observer is capable of forming a reliable judgment of the quality under investigation; and if not, is the fault his, or is it due to the fact that he has been asked to perform an impossible task?

(*b*) Is there any significant concordance of preferences between observers?

(*c*) Can the quality under discussion be represented by a linear variable?

4. The method of offering for judgment objects two at a time is known as the method of paired comparisons. Hitherto it has been used mainly in human psychology, but it has some interesting applications in animal experimentation. For instance, in feeding experiments it is impossible to get an animal to rank a number of foods in order of preference but it is not difficult to offer pairs of foods and to note which is taken first. Experiments of this kind have, of course, to be conducted with great care to ensure that conditions operating when the different pairs are offered are as constant as possible; but the difficulties are far from being insuperable and the method of paired comparisons offers a useful technique in cases where the more usual procedures cannot be applied. From the point of view of theoretical statistics perhaps the most interesting part of the present work is that it offers some lines of approach to the difficult question whether a given quality can be legitimately regarded as based on a linear variable, i.e. whether ranking or scoring methods are justifiable or not.

CONSISTENCE IN PREFERENCES

5. If the object A is preferred to B we write $A \rightarrow B$ or $B \leftarrow A$. The $\binom{n}{2}$ preferences of a single observer may be represented in tabular form as shown in Table I.

In this table, which is shown for the six objects A to F, an entry of unity in column Y and row X means $X \rightarrow Y$, and is thus accompanied by a complementary zero in row Y and column X. The diagonals are blocked out. For example, in Table I, $A \rightarrow B$, $A \rightarrow C$, $D \rightarrow A$, etc.

The arrangement of the objects A to F in the row and column headings is quite arbitrary. There are $(n\,!)^2$ ways of representing the same configuration of preferences in such a table according to the permutations of objects in row and

* That is, we exclude cases in which an observer cannot make up his mind which object he prefers, just as in the ranking case one excludes the possibility of split ranks. In practice it sometimes happens that an observer is genuinely unable to reach a decision. To allow for this fact in the theoretical discussions would introduce complications of a most intractable kind. When the effect becomes important in practice it can be allowed for by selecting the set of preferences which are most unfavourable to the hypothesis under test.

TABLE I

	A	*B*	*C*	*D*	*E*	*F*
A	—	1	1	0	1	1
B	0	—	0	1	1	0
C	0	1	—	1	1	1
D	1	0	0	—	0	0
E	0	0	0	1	—	1
F	0	1	0	1	0	—

column; but in practice it is generally desirable to have the order in row and column the same, and even among the $n!$ possible arrangements so given there are often practical considerations which determine one order as more convenient than others.

6. Paired comparisons may also be represented geometrically by a method which can be illustrated for the case of the six objects as follows:

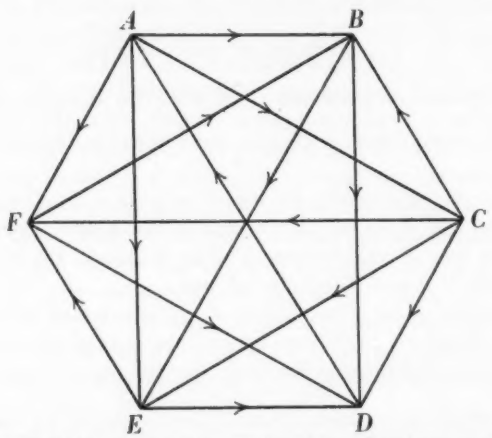

Fig. 1. Geometrical representations of the scheme of preferences of Table I.

We represent the six objects A to F by the six vertices of a regular hexagon and join the vertices in all possible ways by straight lines. If $A \rightarrow B$ we draw

an arrow on the line AB pointing from A to B. The arrows shown on Fig. 1 correspond to the preferences shown in Table I.*

7. If an observer makes preferences of type $A \rightarrow B \rightarrow C \rightarrow A$ we say that the triad ABC is inconsistent. In the geometrical representation an inconsistent triad is shown by a triangle in which all the arrows go round in the same direction. We may thus speak of a "circular" triad of preferences. In Fig. 1 the triads ACD, BEF and three others are circular.

It is also possible to have inconsistent triads of greater extent; but any such circuit must contain at least two circular triads. Suppose, for instance, that $ABCD$ is circular, e.g. that $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$. Then either $A \rightarrow C$ or $C \rightarrow A$. In the first case ACD is circular, in the second ABC. Similarly either ABD or BCD is circular. Thus the circular tetrad must contain just two circular triads. On the other hand it is possible for a tetrad to contain circular triads without being itself circular.

Similarly, if $ABCDE$ is circular either ABC or $ACDE$ is circular and either BCD or $BDEA$ is circular. If the two tetrads are circular there must be at least three circular triads (not necessarily four, because ADE may be common to both). It is easy to see by an actual example based on this configuration that there need not be more than three circular triads; and it is clear that there must be at least three. For if the tetrads are not circular then ABC and BCD must be so and then either CDE is circular or $ABCE$ is so, adding at least one more.

Generally, it appears that a circular n-ad must contain at least $(n-2)$ circular triads; but it may contain more, and the fact that an n-ad contains $(n-2)$ circular triads does not mean that it is itself circular. In discussing inconsistences, therefore, it seems best to confine attention to circular triads, which, so to speak, constitute the inconsistent elements of the configuration, and to ignore the more ambiguous criteria associated with circular polyads of greater extent.

8. We now prove the following theorems:

(1) The maximum possible number of circular triads is $(n^3-n)/24$ if n is odd and $(n^3-4n)/24$ if n is even; and the minimum number is zero.

(2) These limits can always be attained by some configuration of preferences.

(3) For any integral number between the maximum and the minimum there exists at least one preference-configuration with that number of circular triads; and in general there will be more than one.

Consider a polygon of the type shown in Fig. 1 with n vertices. There will

* These preferences were obtained in an experiment on a dog, which was offered the following foods in pairs: meat, biscuit, chocolate, apple, pear and cheese. The members of a pair were cut to the same size and placed equidistantly from the dog, which was then released and allowed to choose. All the pieces of food were eaten avidly, it being that sort of dog, but there were considerable inconsistences in choice. We do not offer these data as more than an illustration of the method.

be $(n-1)$ lines emanating from each vertex. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the number of lines at the respective vertices on which the arrows *leave* the vertex.

Then $$\sum_{r=1}^{n} (\alpha_r) = \binom{n}{2}$$

and the mean value of α_r is $(n-1)/2$.

Define $$T = \sum_{r=1}^{n} \left(\alpha_r - \frac{n-1}{2}\right)^2$$

$$= S(\alpha_r^2) - \frac{n(n-1)^2}{4}. \qquad \ldots\ldots(1)$$

We now show that if the direction of a preference is altered and the effect is to increase the number of circular triads by d, T is reduced by $2d$; and conversely. Consider the preference $A \to B$. The only triads affected by altering this to $B \to A$ are those containing the line AB. Suppose there are α preferences of type $A \to X$ (including $A \to B$) and β preferences of type $B \to X$. Then four possible types of triad arise:

$A \to X \leftarrow B$, say p in number
$A \leftarrow X \to B$,
$A \to X \to B$, which must number $\alpha - p - 1$
$A \leftarrow X \leftarrow B$, ,, ,, $\beta - p$.

When the preference $A \to B$ is reversed the first two remain non-circular. The third becomes circular, the fourth ceases to be so. The reduction in the value of T is

$$\alpha^2 - (\alpha-1)^2 + \beta^2 - (\beta+1)^2$$
$$= 2(\alpha - \beta - 1)$$
$$= 2d, \text{ say.}$$

The increase in the number of circular triads is

$$(\alpha - p - 1) - (\beta - p) = \alpha - \beta - 1$$
$$= d.$$

More generally, if as the result of reversing any number of preferences T is decreased by $2d$, then d must be an integer and the number of circular triads must be increased by d. This clearly follows from the previous results for the reversal of preferences can take place one at a time and the effect on T and the number of circular triads is cumulative.

We now investigate the maximum and minimum values of T. It is clear from the definition that T is greatest when the α's are the natural numbers $1, 2, \ldots, n$; and this is a possible case because it corresponds to ordinary ranking. Hence max. $(T) = (n^3 - n)/12$.

For the minimum value, consider the polygon $A_1, A_2, \ldots, A_n$. Set up the preferences $A_1 \to A_2 \to \ldots A_n \to A_1$. Clearly at any vertex this results in one arrow

entering and one leaving the vertex, i.e. the contribution to α is unity at each vertex. Next set up the preferences $A_1 \to A_3 \to A_5 \to \ldots$. This circuit may either visit each vertex once, or not. In the latter case we proceed to an unvisited vertex and set up the preferences $A_r \to A_{r+2} \to A_{r+4} \to \ldots$ and so on. Again there will be a unit contribution to all the α's.

We then set up the preferences $A_1 \to A_4 \to A_7 \to$ etc. and so on; and in this way we shall ultimately complete the preference scheme.

If n is odd all the preferences described will consist of circular tours of the polygon, and thus the value of α for each vertex will be $(n-1)/2$. If n is even the last preference $A_1 \to A_{\frac{1}{2}n+1}$ will not be a tour but will consist of the single line joining one vertex with the symmetrically opposite vertex. Thus there will be $n/2$ vertices for which $\alpha = n/2$ and $n/2$ vertices for which $\alpha = (n-2)/2$. In this case $T = n/4$.

Now it is clear from the definition of T that it cannot be less than zero, or if n is even, be less than $n/4$. The configuration just given shows that these minima are, in fact, attainable.

Thus T can vary from a maximum of $(n^3-n)/12$ to a minimum of zero or $n/4$. Hence the maximum number of circular triads, being half the variation from maximum to minimum of T (the maximum of T corresponding to the ranking case in which there are no inconsistences) is $(n^3-4n)/24$ if n is even and $(n^3-n)/24$ if n is odd.

This establishes the first two results enunciated at the beginning of this section. To prove the third it is sufficient to give a systematic method of proceeding from the configuration of minimum to that of maximum inconsistence by steps decreasing T two at a time. Consider, for example, the case $n = 8$. For the minimum inconsistence the α's will have the values 0 to 7, which we set out thus:

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
0	1	2	3	4	5	6	7

We proceed by reversing the preferences between vertices whose α-values differ by two. This clearly reduces T by two.

Reversing the preferences between C and E we get

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
0	1	3	3	3	5	6	7

and between D and F we get

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
0	1	3	4	3	4	6	7

which we may rearrange as

A	*B*	*C*	*E*	*D*	*F*	*G*	*H*
0	1	3	3	4	4	6	7

Now reversing the preferences between B and E and between D and G and rearranging we have

A	B	E	C	F	D	G	H
0	2	2	3	4	5	5	7

and now interchanging A and B, G and H,

A	B	E	C	F	D	G	H
1	1	2	3	4	5	6	6

At this stage we have preserved the α-numbers 2, 3, 4 and 5 in the middle but reduced the extremes A and H. We can now carry out the process again, arriving at the α-numbers.

1 2 2 3 4 5 5 6

and twice again, giving

2 2 3 3 4 4 5 5

whence a final interchange gives

3 3 3 3 4 4 4 4

and this is the position of maximum inconsistence. It is readily verified by following the interchanges on a polygon diagram that the reversals are, in fact, legitimate.

Coefficient of consistence in paired comparisons

9. If d is the number of circular triads in an observed configuration of preferences we define

$$\left.\begin{aligned} \zeta &= 1-\frac{24d}{n^3-n}, \quad n \text{ odd} \\ &= 1-\frac{24d}{n^3-4n}, \quad n \text{ even} \end{aligned}\right\} \qquad \ldots\ldots(2)$$

and call ζ the coefficient of consistence. If and only if it is unity there are no inconsistences in the configuration, which may therefore be represented by a ranking. As ζ decreases to zero the inconsistence, as measured by the number of circular triads, increases.

For example, in the configuration of Fig. 1 there are five circular triads, *ABD*, *ACD*, *AFD*, *AED* and *BEF*. The maximum possible number is 8. Thus $\zeta = 0{\cdot}375$.

10. ζ can also be interpreted in the light of Table I. Suppose, in that table, we sum the rows. (The column sums are determined by the row sums and add no fresh information.) The sum of any row will be the α-number for that vertex in the polygon which corresponds to the object defining the row. T will then be the value of the sum of squares of deviations of row totals from the mean value $(n-1)/2$, that is to say, will be the variance of the row sums multiplied

by n. ζ is thus a linear function of this variance; but it cannot be tested in the χ^2 distribution as if Table I were a contingency table, for the border cells are not independent or linearly dependent.

11. If an individual observer produces a configuration of preferences which show inconsistence there are usually several explanations; he may be an incompetent judge, the objects may be so alike that consistent differentiation is not possible, or his attention may wander during the course of the experiment. We discuss these questions later. They are mentioned here to explain the motive for the next stage of the mathematics. With what probability can a value of ζ arise by chance if the observer allots his preferences at random with respect to the quality under consideration?

With n objects there are $2^{\binom{n}{2}}$ possible configurations of preferences. We proceed to investigate the distribution of d in this universe of $2^{\binom{n}{2}}$ different members. The method consists of proceeding from the distribution for n to that for $(n+1)$.

For $n = 3$ there are eight configurations, of which two give one circular triad and six no circular triads. Consider the effect of adding a new vertex D to the vertices ABC. Four cases arise:

(1) $D\rightarrow$ all A, B, C:
(2) $D\rightarrow$ two of A, B, C.
(3) $D\rightarrow$ one of A, B, C.
(4) $D\rightarrow$ none of A, B, C.

The last two are symmetrical with the first two and need not be separately considered.

Situation (1) arises in one way and clearly does not add any new circular triads other than those already existing in the configuration ABC. It therefore contributes six values $d = 0$ and two values $d = 1$. So does situation (4).

Situation (2) arises in three ways, according as $D\leftarrow A$, B, or C. The configurations so reached are similar and we may take any one, say $D\leftarrow C$, as the single preference. If $A\leftarrow C$ then DAC is not circular and if $B\leftarrow C$ the DBC is not circular. On the other hand $A\rightarrow C$ and $B\rightarrow C$ will each produce a circular triad. We then have the cases

	No. of circular triplets added
$A\leftarrow C\rightarrow B$	0
$A\rightarrow C\rightarrow B$	1
$A\leftarrow C\leftarrow B$	1
$A\rightarrow C\leftarrow B$	2

We now consider AB. In the first two cases just enumerated the direction of AB does not matter and no circular triads are added. With the third $A \rightarrow B$ gives no circular triad but $A \leftarrow B$ adds one. With the fourth $A \rightarrow B$ adds one and $A \leftarrow B$ adds none.

Thus the number of circular triads occurring for these four cases is found to be

No. of circular triplets	Frequency
0	2
1	2
2	4

We must multiply the frequency by three and by two to allow for similar symmetrical arrangements, and the final results are

No. of circular triplets	Frequency
0	24
1	16
2	24
Total	64

The principles of this method are clear enough and the work may be formalized by a number of conventions which we omit to save space. In common with many similar combinatorial problems, however, troubles arise from the sheer number of possibilities and the difficulty of ensuring that nothing is overlooked. Up to the present we have found the distribution of d for n up to and including 7. The frequencies and probabilities are given in Table II.

12. For the values already obtained the moments are given by the following formulae:

$$\mu_1' \text{ (about 0)} = \frac{1}{4}\binom{n}{3}, \qquad \text{.....(3)}$$

$$\mu_2 = \frac{3}{16}\binom{n}{3}, \qquad \text{.....(4)}$$

$$\mu_3 = -\frac{3}{32}\binom{n}{3}(n-4), \qquad \text{.....(5)}$$

$$\mu_4 = \frac{3}{256}\binom{n}{3}\left\{9\binom{n-3}{3} + 39\binom{n-3}{2} + 9\binom{n-3}{1} + 7\right\}. \qquad \text{.....(6)}$$

We have very little doubt that these results are true in general but can offer no rigorous proof. In so far, however, as the moments are in a sense symmetric sums it appears highly probable that they are given by polynomials

TABLE II

Frequency (f) of values of d and probability (P) that values will be attained or exceeded

Value of d	$n=2$		$n=3$		$n=4$		$n=5$		$n=6$		$n=7$	
	f	P	f	P	f	P	f	P	f	P	f	P
0	2	1·000	6	1·000	24	1·000	120	1·000	720	1·000	5,040	1·000
1			2	0·250	16	·625	120	·883	960	·978	8,400	·998
2					24	·375	240	·766	2,240	·949	21,840	·994
3							240	·531	2,880	·880	33,600	·983
4							280	·297	6,240	·792	75,600	·967
5							24	·023	3,648	·602	90,384	·931
6									8,640	·491	179,760	·888
7									4,800	·227	188,160	·802
8									2,640	·081	277,200	·713
9											280,560	·580
10											384,048	·447
11											244,160	·263
12											233,520	·147
13											72,240	·036
14											2,640	·001
Total	2	—	8	—	64	—	1,024	—	32,768	—	2,097,152	—

in n; and if this is so the values obtained are sufficient to establish polynomials of degree six or less.

It is also to be noted that from the above values of the moments

$$\beta_1 = \mu_3^2/\mu_2^3 \sim 8/n, \qquad \beta_2 = \mu_4/\mu_2^2 \sim 3 + 12/n,$$

from which it appears that a Type III distribution would fit the d-distribution fairly closely for moderate or large values of n. But as the distribution of d is of interest mainly for low values of n, which are all that occur in practice, it hardly seems worth while attempting to fit a curve.

AGREEMENT AMONG SEVERAL OBSERVERS

13. We now consider the investigation of similarities of judgments for m observers. Suppose that in a table of the form of Table I we enter a unit in the cell in row X and column Y whenever $X \to Y$ and count the units in each cell. A cell may then contain any number from 0 to m. If the observers are in complete agreement there will be $\binom{n}{2}$ cells containing the number m, the remaining $\binom{n}{2}$ cells being zero. The agreement may be complete even if there are inconsistences present.

Suppose that the cell in row X and column Y contains the number γ. Let

$$\Sigma = S\binom{\gamma}{2}, \qquad \text{.....(7)}$$

the summation extending over the $n(n-1)$ cells of the table (the diagonal cells being ignored). Σ is then the sum of the number of agreements between pairs of judges. Put

$$u = \frac{2\Sigma}{\binom{m}{2}\binom{n}{2}} - 1. \qquad \text{.....(8)}$$

The maximum number of agreements, occurring if $\binom{n}{2}$ cells each contain m, is $\binom{n}{2}\binom{m}{2}$ and thus in the case of complete agreement, and only in this case, $u = 1$. The further we go from this case, as measured by agreements between pairs of observers, the smaller u becomes. The minimum number of agreements occurs when each cell contains $m/2$ if m is even or $(m \pm 1)/2$ if m is odd. That is, if m is even, the minimum number of agreements is

$$2\binom{\frac{m}{2}}{2}\binom{n}{2} = \tfrac{1}{4}m(m-2)\binom{n}{2},$$

and in this case

$$u = -\frac{1}{m-1}. \qquad \text{.....(9)}$$

When m is odd the minimum value of u is found to be

$$u = -\frac{1}{m}. \qquad \text{.....(10)}$$

14. We propose to call u the coefficient of agreement. It is unity if and only if there is complete agreement in the comparisons. Its minimum value is not -1 except when $m = 2$. This, however, is to be expected in a measure of agreement for there can be no such thing as complete disagreement among three or more observers in paired comparisons. If observer P differs in certain comparisons from observers Q and R, the two latter must agree on those comparisons.

When $m = 2$, u reduces to

$$u = \frac{2\Sigma}{\binom{n}{2}} - 1 \qquad \text{.....(11)}$$

and Σ becomes twice the number of cases in which the two observers agree about a comparison. u is thus a generalization of a coefficient τ proposed by Kendall (1938) to measure the correlation between two rankings. For general m, if the entries in the table were constrained to the ranking type, u would be the average intercorrelation τ between observers taken two at a time.

15. In discussing the significance of u it is desirable to know whether the set of preferences which give rise to it could have arisen by chance if the preferences had been assigned at random with respect to the quality under consideration. The procedure which first suggests itself is a generalization of the method used for the case of m rankings (Kendall & Babington Smith, 1939). That is to say, we sum the entries in the rows of the table and consider the variance of these entries. If the preferences are allotted at random we expect to find about equal numbers given to each object, and the variance will be low; in other cases it will be higher.

The difficulty about this suggestion is that it has not been found possible to ascertain the distribution of the variance in the $2^{m\binom{n}{2}}$ possible sets of preferences. The case $m = 1$, corresponding to the distribution of d for inconsistences, is difficult enough to solve. For higher values of m we have failed to find any distributions except in trivial cases.

We can, however, offer a test based on the distribution of u (or Σ). The comparative simplicity of the distributions in this case is in accordance with the remark made by Kendall in the paper under reference that the distribution of τ is much simpler and much more regular than the distribution of the Spearman correlation coefficient ρ.

16. Consider one cell in the table in row X and column Y and let it contain the number γ. Then the corresponding cell in row Y and column X will contain $m-\gamma$. Thus these two contribute to Σ the amount $\binom{\gamma}{2}+\binom{m-\gamma}{2}$.

Now, of the total ways in which the units can be distributed in the first cell there will be $\binom{m}{\gamma}$ in which γ units occur. Consequently the distribution of Σ in the cell and the corresponding cell is given by the expression

$$f = t^{\binom{m}{2}} + \binom{m}{1} t^{\binom{m-1}{2}} + \binom{m}{2} t^{\binom{m-2}{2}+\binom{2}{2}} + \ldots + \binom{m}{\gamma} t^{\binom{m-\gamma}{2}+\binom{\gamma}{2}} + \ldots + t^{\binom{m}{2}}, \quad \ldots\ldots(12)$$

and since the distribution in other pairs of cells is independent if the preferences are allotted at random the distribution of Σ for the whole table is given by

$$D(\Sigma) = f^N, \quad \ldots\ldots(13)$$

where $N = \binom{n}{2}$.

17. The distributions have been worked out for the following values of m and n: $m = 3$, $n = 2$ to 8; $m = 4$, $n = 2$ to 6; $m = 5$, $n = 2$ to 5; $m = 6$, $n = 2$ to 4. Tables III to VI give the probabilities based on these distributions, i.e. the probabilities that a given value of Σ will be attained or exceeded.

For constant n the distribution tends to the Type III form as m tends to infinity. In fact, for a single pair of related cells the variate value corresponding

TABLE III

The probability P that a value of Σ will be attained or exceeded, for m = 3, n = 2 to 8

n=2		n=3		n=4		n=5		n=6		n=7		n=8	
Σ	P	Σ	P	Σ	P	Σ	P	Σ	P	Σ	P	Σ	P
1	1·000	3	1·000	6	1·000	10	1·000	15	1·000	21	1·000	28	1·000
3	·250	5	·578	8	·822	12	·944	17	·987	23	·998	30	1·000
		7	·156	10	·466	14	·756	19	·920	25	·981	32	·997
		9	·016	12	·169	16	·474	21	·764	27	·925	34	·983
				14	·038	18	·224	23	·539	29	·808	36	·945
				16	·0046	20	·078	25	·314	31	·633	38	·865
				18	$\cdot 0^{3}24$	22	·020	27	·148	33	·433	40	·736
						24	·0035	29	·057	35	·256	42	·572
						26	$\cdot 0^{3}42$	31	·017	37	·130	44	·400
						28	$\cdot 0^{4}30$	33	·0042	39	·056	46	·250
						30	$\cdot 0^{6}95$	35	$\cdot 0^{3}79$	41	·021	48	·138
								37	$\cdot 0^{3}12$	43	·0064	50	·068
								39	$\cdot 0^{4}12$	45	·0017	52	·029
								41	$\cdot 0^{6}92$	47	$\cdot 0^{3}37$	54	·011
								43	$\cdot 0^{7}43$	49	$\cdot 0^{4}68$	56	·0038
								45	$\cdot 0^{8}93$	51	$\cdot 0^{4}10$	58	·0011
										53	$\cdot 0^{5}12$	60	$\cdot 0^{3}29$
										55	$\cdot 0^{6}12$	62	$\cdot 0^{4}66$
										57	$\cdot 0^{8}86$	64	$\cdot 0^{4}13$
										59	$\cdot 0^{9}44$	66	$\cdot 0^{5}22$
										61	$\cdot 0^{10}15$	68	$\cdot 0^{6}32$
										63	$\cdot 0^{12}23$	70	$\cdot 0^{7}40$
												72	$\cdot 0^{8}42$
												74	$\cdot 0^{9}36$
												76	$\cdot 0^{10}24$
												78	$\cdot 0^{11}13$
												80	$\cdot 0^{13}48$
												82	$\cdot 0^{14}12$
												84	$\cdot 0^{16}14$

TABLE IV

The probability P that a value of Σ will be attained or exceeded, for m = 4 and n = 2 to 6 (for n = 6 only values beyond the 1% point are given)

n=2		n=3		n=4		n=5		n=5		n=6		n=6	
Σ	P	Σ	P	Σ	P	Σ	P	Σ	P	Σ	P	Σ	P
2	1·000	6	1·000	12	1·000	20	1·000	42	·0048	57	·014	79	$\cdot 0^{8}42$
3	·625	7	·947	13	·997	21	1·000	43	·0030	58	·0092	80	$\cdot 0^{8}28$
6	·125	8	·736	14	·975	22	·999	44	·0017	59	·0058	81	$\cdot 0^{9}98$
		9	·455	15	·901	23	·995	45	$\cdot 0^{3}73$	60	·0037	82	$\cdot 0^{9}15$
		10	·330	16	·769	24	·979	46	$\cdot 0^{3}41$	61	·0022	83	$\cdot 0^{9}12$
		11	·277	17	·632	25	·942	47	$\cdot 0^{3}24$	62	·0013	84	$\cdot 0^{10}51$
		12	·137	18	·524	26	·882	48	$\cdot 0^{4}90$	63	$\cdot 0^{3}76$	86	$\cdot 0^{11}30$
		14	·043	19	·410	27	·805	49	$\cdot 0^{4}37$	64	$\cdot 0^{3}44$	87	$\cdot 0^{11}17$
		15	·025	20	·278	28	·719	50	$\cdot 0^{4}25$	65	$\cdot 0^{3}23$	90	$\cdot 0^{13}28$
		18	·0020	21	·185	29	·621	51	$\cdot 0^{5}93$	66	$\cdot 0^{3}13$		
				22	·137	30	·514	52	$\cdot 0^{6}21$	67	$\cdot 0^{4}72$		
				23	·088	31	·413	53	$\cdot 0^{5}17$	68	$\cdot 0^{4}36$		
				24	·044	32	·327	54	$\cdot 0^{6}74$	69	$\cdot 0^{4}18$		
				25	·027	33	·249	56	$\cdot 0^{7}66$	70	$\cdot 0^{5}97$		
				26	·019	34	·179	57	$\cdot 0^{7}38$	71	$\cdot 0^{5}47$		
				27	·0079	35	·127	60	$\cdot 0^{9}93$	72	$\cdot 0^{5}20$		
				28	·0030	36	·090			73	$\cdot 0^{5}10$		
				29	·0025	37	·060			74	$\cdot 0^{6}51$		
				30	·0011	38	·038			75	$\cdot 0^{6}18$		
				32	$\cdot 0^{3}16$	39	·024			76	$\cdot 0^{7}78$		
				33	$\cdot 0^{4}95$	40	·016			77	$\cdot 0^{7}44$		
				36	$\cdot 0^{5}38$	41	·0088			78	$\cdot 0^{7}15$		

TABLE V

The probability P that a value of Σ will be attained or exceeded, for m = 5 and n = 2 to 5

$n=2$		$n=3$		$n=4$		$n=5$		$n=5$	
Σ	P	Σ	P	Σ	P	Σ	P	Σ	P
4	1·000	12	1·000	24	1·000	40	1·000	76	$\cdot 0^{4}50$
6	·375	14	·756	26	·940	42	·991	78	$\cdot 0^{4}16$
10	·063	16	·390	28	·762	44	·945	80	$\cdot 0^{5}50$
		18	·207	30	·538	46	·843	82	$\cdot 0^{5}15$
		20	·103	32	·353	48	·698	84	$\cdot 0^{6}39$
		22	·030	34	·208	50	·537	86	$\cdot 0^{6}10$
		24	·011	36	·107	52	·384	88	$\cdot 0^{7}23$
		26	·0039	38	·053	54	·254	90	$\cdot 0^{8}53$
		30	$\cdot 0^{3}24$	40	·024	56	·158	92	$\cdot 0^{8}12$
				42	·0093	58	·092	94	$\cdot 0^{9}14$
				44	·0036	60	·050	96	$\cdot 0^{10}46$
				46	·0012	62	·026	100	$\cdot 0^{12}91$
				48	$\cdot 0^{3}36$	64	·012		
				50	$\cdot 0^{3}12$	66	·0057		
				52	$\cdot 0^{4}28$	68	·0025		
				54	$\cdot 0^{5}54$	70	·0010		
				56	$\cdot 0^{5}18$	72	$\cdot 0^{3}39$		
				60	$\cdot 0^{7}60$	74	$\cdot 0^{3}14$		

TABLE VI

The probability P that a value of Σ will be attained or exceeded, for m = 6 and n = 2 to 4

$n=2$		$n=3$		$n=4$		$n=4$		$n=4$	
Σ	P	Σ	P	Σ	P	Σ	P	Σ	P
6	1·000	18	1·000	36	1·000	55	·043	74	$\cdot 0^{4}12$
7	·688	19	·969	37	·999	56	·029	75	$\cdot 0^{5}89$
10	·219	20	·832	38	·991	57	·020	76	$\cdot 0^{5}49$
15	·031	21	·626	39	·959	58	·016	77	$\cdot 0^{5}32$
		22	·523	40	·896	59	·011	80	$\cdot 0^{6}68$
		23	·468	41	·822	60	·0072	81	$\cdot 0^{6}17$
		24	·303	42	·755	61	·0049	82	$\cdot 0^{6}12$
		26	·180	43	·669	62	·0034	85	$\cdot 0^{7}34$
		27	·147	44	·556	63	·0025	90	$\cdot 0^{8}93$
		28	·088	45	·466	64	·0016		
		29	·061	46	·409	65	$\cdot 0^{3}83$		
		30	·040	47	·337	66	$\cdot 0^{3}66$		
		31	·034	48	·257	67	$\cdot 0^{3}48$		
		32	·023	49	·209	68	$\cdot 0^{3}26$		
		35	·0062	50	·175	69	$\cdot 0^{3}16$		
		36	·0029	51	·133	70	$\cdot 0^{4}86$		
		37	·0020	52	·097	71	$\cdot 0^{4}68$		
		40	$\cdot 0^{3}58$	53	·073	72	$\cdot 0^{4}48$		
		45	$\cdot 0^{4}31$	54	·057	73	$\cdot 0^{4}16$		

to a frequency $\binom{m}{\gamma}$ is $\binom{m-\gamma}{2}+\binom{\gamma}{2}$, which is a quadratic in γ. Were the variate value a linear function of γ the distribution for the single cell would tend to normality in accordance with the well-known property of the binomial. The case of the quadratic value corresponds to a transformation of the variate of the type $x^2 = y$ and the transform of the normal form $\exp(-x^2)\,dx$ becomes the Type III form $\exp(-y)\,y^{-\frac{1}{2}}dy$. Since the N cells are independent and the sum of variates in the same Type III form is also distributed in that form, it follows that Σ is in the limit distributed as $\exp(-\Sigma)\,\Sigma^{\frac{N}{2}-1}d\Sigma$ except perhaps for some constants. Thus Σ or some multiple of it is distributed as χ^2.

For constant m the distribution tends to normality with increasing n.

18. The first of these results suggests that the Type III distribution will provide an approximation to the distribution (13) when m is moderately large. We proceed to find the first four moments of (13).

It is sufficient to find the first four moments of (12), those of (13) being obtainable therefrom in virtue of the relationships which connect seminvariants of independent distributions.

The rth moment of (12) about the origin is given by

$$2^m\mu_r' = \left[\left(t\frac{\partial}{\partial t}\right)^r f\right]_{t=1}, \qquad \text{.....(14)}$$

since 2^m is the total frequency. Thus we have

$$2^m\mu_1' = \sum_{r=0}^{m}\binom{m}{r}\left(r^2 - mr + \frac{m^2-m}{2}\right) = 2^m\binom{m}{2} + S\binom{m}{r}(r^2-mr). \qquad \text{.....(15)}$$

Sums such as $S\binom{m}{r}r^p$ can be obtained by operating on the binomial $(1+x)^m$ p times by $x\frac{\partial}{\partial x}$, e.g. we find

$$S\left\{\binom{m}{r}r\right\} = 2^m\frac{m}{2},$$

$$S\binom{m}{r}r^2 = 2^m\left\{\frac{m}{2}+\frac{1}{2}\binom{m}{2}\right\}$$

and hence, substituting in (15),

$$\mu_1' = \frac{1}{2}\binom{m}{2}.$$

Thus the mean of the distribution (13) is given by

$$\mu_1' = \tfrac{1}{2}N\binom{m}{2}. \qquad \text{.....(16)}$$

In a similar way we find

$$\mu_2 = \tfrac{1}{4}N\binom{m}{2}, \qquad \text{.....(17)}$$

$$\mu_3 = \tfrac{3}{4}N\binom{m}{3}, \qquad \text{.....(18)}$$

$$\mu_4 = N\binom{m}{2}\left\{\frac{3m^2-15m+17}{8} + \tfrac{3}{32}N(m^2-m)\right\}. \qquad \text{.....(19)}$$

These are the moments of Σ. Those of u are obtained by dividing by an appropriate power of $N\binom{m}{2}$ and it may be noted in particular that the mean of u is zero.

We have directly from (17), (18) and (19)

$$\beta_1 = \frac{8}{N}\frac{(m-2)^2}{m(m-1)},$$

$$\beta_2 = \frac{4}{Nm(m-1)}\left\{3m^2-15m+17+\frac{3N}{4}m(m-1)\right\}.$$

For constant m, as $N\to\infty$,

$$\beta_1\to 0, \quad \beta_2\to 3$$

and for constant N, as $m\to\infty$,

$$\beta_1\to\frac{8}{N}, \quad \beta_2\to\frac{12}{N}+3,$$

confirming the tendency towards the Type III distribution.

19. The first four moments of the Type III distribution

$$dF = ke^{-px}\,x^{q-1}\,dx$$

are

$$\frac{q}{p}, \frac{q}{p^2}, \frac{2q}{p^3}, \frac{3q(q+2)}{p^4}.$$

Equating the second and third moments to those given by (17) and (18) we find

$$q = \frac{Nm(m-1)}{2(m-2)^2}, \qquad \text{.....(20)}$$

$$p = \frac{2}{m-2}. \qquad \text{.....(21)}$$

To make the first moments correspond we move the origin of the Σ dis-

tribution a distance $\frac{1}{2}N\binom{m}{2}\frac{m-3}{m-2}$ to the right. We thus reach the approximation to the Σ distribution, coinciding in the first three moments

$$dF = ke^{-\frac{2x}{m-2}}x^{\frac{Nm(m-1)}{2(m-1)^2}-1}dx,$$

where

$$x = \Sigma - \tfrac{1}{2}N\binom{m}{2}\frac{m-3}{m-2}$$

or, transforming to the more usual χ^2 form by putting $\chi^2 = 4x/(m-2)$, we find that

$$\left\{\Sigma - \tfrac{1}{2}N\binom{m}{2}\frac{m-3}{m-2}\right\}\frac{4}{m-2} \qquad \text{.....(22)}$$

is distributed as χ^2 with

$$\nu = \frac{Nm(m-1)}{(m-2)^2} \qquad \text{.....(23)}$$

degrees of freedom.

The fourth moments of Σ and the χ^2 approximation differ by terms of order N^{-1} and m^{-1} compared with their absolute values.

20. It only remains to be seen how large m and n must be for this to provide a satisfactory approximation.

Consider first the distributions for $m = 3$. When $n = 8$, $N = 28$, we have, for the approximation, 4Σ distributed with 168 degrees of freedom. From Table III we see that for $\Sigma = 54$, $P = 0{\cdot}011$ and for $\Sigma = 58$, $P = 0{\cdot}0011$. Applying a continuity correction by deducting unity from Σ we find for the χ^2 approximation with $\chi^2 = 4 \times 53$, $\nu = 168$, $P = 0{\cdot}011$, and with $\chi^2 = 4 \times 57$, $P = 0{\cdot}00114$. The correspondence is very close, in spite of the low value of m.

For $m = 4$, $n = 5$, $N = 10$, the approximation gives $2\Sigma - 30$ distributed with 30 degrees of freedom. For $\Sigma = 40$ and 41, this gives, with continuity corrections of 0·5, half the variate-interval, $\chi^2 = 49$ and 51, $\nu = 30$. From the diagram given in Yule & Kendall's "Introduction to the Theory of Statistics" (1937) it is seen that these values lie one on either side of the 1% value; and this is in accordance with the exact values of P, which are seen from Table IV to be 0·016 and 0·0088. Similarly we find that the values of Σ, 37 and 38, lie on either side of the 5% level, which is again in accordance with the exact values, $P = 0{\cdot}060$ and 0·038.

For $m = 6$, $n = 4$, $N = 6$, the approximation gives $\Sigma - 33{\cdot}75$ distributed with 11·25 degrees of freedom. For $\Sigma = 59$ and 60 the corresponding χ^2 values are seen to lie on either side of the 1% point, which accords with the exact value of Table VI.

We conclude that the χ^2 approximation provides an adequate test of significance for the values of m and n outside the range for which Tables III–VI give exact values.

21. As a matter of theoretical interest we may record the results for the distribution of u when the data are ranked. It appears that in this case

$$\frac{12}{m-2}\frac{2n+5}{2n^2+6n+7}\left[\Sigma - \frac{1}{2}\binom{m}{2}\binom{n}{2}\left\{1-\frac{1}{3(m-2)}\frac{(2n+5)^2}{2n^2+6n+7}\right\}\right]$$

is distributed approximately as χ^2 with

$$\binom{m}{2}\frac{2}{(m-2)^2}\binom{n}{2}\frac{(2n+5)^3}{(2n^2+6n+7)} \text{ degrees of freedom.}$$

This result is not of much practical value. The case of m rankings can be more simply treated by other methods.

INTERPRETATION OF RESULTS OF PAIRED COMPARISONS

22. In the light of the foregoing theory we may discuss the interpretation of the results of a paired comparison experiment.

If for each observer the coefficient of consistence is unity the comparisons reduce to rankings and may be discussed by known methods. But if some or all the coefficients are not unity we have to consider the following possibilities:

(*a*) Some of the observers may be bad judges and the inconsistences reflect their shortcomings in making comparisons.

(*b*) Some of the objects may differ by amounts which fall below the threshold of distinguishability for some observers.

(*c*) The property under judgment may not be a linear variate at all and we may be getting the sort of confusion which would result if observers were asked to compare English towns according to the bivariate concept "geographical position".

(*d*) Several of the effects may be operating simultaneously.

23. If we have only one observer and have no prior knowledge of his capabilities it is not in general possible to apportion his inconsistences among these causes. Exceptions may occur when the inconsistences are of a marked and peculiar kind; for instance if they involve only four objects out of 15, we may suspect that the four are practically indistinguishable rather than that the observer is unable to make distinctions at all and avoided inconsistences among the others by sheer chance. But even here conclusions drawn *a posteriori* after inspection of the data are dubious. Table II gives a test of the hypothesis that an observer is incapable of making judgments. For example, with $n = 7$, the chances are 983 in 1000 that if the preferences are made at random there will be more than two inconsistent triads, so that if we find two or less, it is improbable that the observer is completely incapable of judgment. We might then be led to suppose that his small deviation from internal consistence is due to fluctuation of attention, very close resemblance to the objects giving rise to the inconsistences, or both.

24. With m observers the investigation can be taken a good deal further. If all the observers show inconsistences we suspect that the objects are at fault or that the observers are being asked to perform an impossible task. On the other hand, if most of the observers show a small or zero inconsistence we suspect that the others are just bad judges and may reject their data accordingly.

As between indistinguishability of objects and non-linearity of variate, a choice of explanations would depend largely on the extent to which inconsistences were concerned with the same set of objects. If there is a high value of u, indicating concordance of judgment, we expect to find most of the inconsistences confined to certain objects, and common to observers. In this case we suspect that the objects are close together in the degree to which they exhibit the quality under consideration. But if the observers scatter their inconsistences over the whole field u will be moderate or low and we suspect that the observers are being asked to do something beyond their capacity; and this brings us to question the validity of regarding the quality as a linear variable.

25. When a quality such as "bravery" or "intelligence" is insusceptible to measurement there is frequently doubt of this kind. But this has not deterred investigators from assuming that such statistical variables exist, or from requiring observers to rank objects according to them, or in some cases from replacing such rankings by quantiles of the normal curve. We are never tired of criticizing this Principle of the Hypostasis of Plausible Terminology. Hitherto it has flourished largely because of the difficulty of adducing evidence against it; and we hope that the inconsistence of paired comparisons will provide a criterion, however rough, of the legitimacy of the methods to which it leads.

But we would emphasize that our approach to the method of paired comparisons has a somewhat different object from that elaborated by Thurstone (1927 and many subsequent papers). As we understand it, his method is appropriate where one is entitled to assume *a priori* or by reason of precautions taken in the selection of material that a linear variable is involved and that there exist perceptible differences between the items presented for comparison. Our object is to make it possible to dispense with such assumptions and precautions.

26. A few words may be added about the case in which an objective order is known to exist (as, for instance, in judging individuals according to age or weight). In such circumstances the appearance of inconsistences will indicate unreliability of the part of the observer or subliminal differences between objects. A measure of the observer's reliability may be obtained by calculating u between known and observed comparisons. If ζ is high enough to enable us to accept his judgments as internally consistent on the whole, u may still be low enough to reject his judgments as accurate.

27. We conclude with an example of the application of the foregoing theory to some experimental material.

Classes of children (ages 11 to 13 inclusive) were asked to state their preferences with respect to certain school subjects. Each child was given a sheet on which were written the possible pairs of subjects and asked to underline the one preferred in each case. Two classes gave the following results:

(*a*) 21 boys, 13 school subjects. The preferences are shown in Table VII, which is in the form described in section 13; e.g. there were 18 boys who preferred Art to Religion.

TABLE VII

Preferences of 21 *boys in* 13 *subjects*

	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
1. Woodwork	—	14	20	15	15	16	16	18	18	18	20	21	20	211
2. Gymnastics	7	—	14	12	13	18	14	16	16	20	16	18	19	183
3. Art	1	7	—	10	14	10	16	18	16	16	17	16	19	160
4. Science	6	9	11	—	11	12	15	14	13	13	17	17	16	154
5. History	6	8	7	10	—	14	11	12	14	15	13	14	16	140
6. Geography	5	3	11	9	7	—	14	14	13	13	16	15	17	137
7. Arithmetic	5	7	5	6	10	7	—	9	11	13	15	13	15	116
8. Religion	3	5	3	7	9	7	12	—	12	14	14	16	14	116
9. English Literature	3	5	5	8	7	8	10	9	—	10	13	13	15	106
10. Commercial subjects	3	1	5	8	6	8	8	7	11	—	10	10	14	91
11. Algebra	1	5	4	4	8	5	6	7	8	11	—	10	13	82
12. English Grammar	0	3	5	4	7	6	8	5	8	11	11	—	13	81
13. Geometry	1	2	2	5	5	4	6	7	6	7	8	8	—	61
													Total	1638

The calculation of Σ for this table, in which the objects are arranged in order of total number of preferences, may be shortened by noting that Σ as given by equation (7) may be transformed into the form

$$\Sigma = S(\gamma^2) - mS(\gamma) + \binom{m}{2}\binom{n}{2},$$

where the summation now takes place over the half of the table below the diagonal. Since the numbers in this half are smaller than those in the other half there is a considerable saving in arithmetic.

We find $$\Sigma = 9718$$

and hence $$u = \frac{2 \times 9718}{\binom{21}{2}\binom{13}{2}} - 1 = 0{\cdot}186.$$

There is thus a certain amount of agreement among the children, indicated by the positive value of u. Is this significant?

We note first of all that this distribution of preferences could not have arisen by chance to any acceptable degree of probability. In fact, $\chi^2 = 412{\cdot}4$ (equation (22)) and $\nu = 90{\cdot}7$. The large value of ν justifies the use of the normal

approximation to the χ^2 distribution and we find $\sqrt{(2\chi^2)} - \sqrt{(2\nu - 1)} = 15{\cdot}3$ a very improbable result on the hypothesis of a random allocation of preferences.

The distribution of circular triads was as follows:

No. of triads	Frequency	No. of triads	Frequency
0	1	12	1
1	1	17	3
4	5	21	1
6	2	25	1
7	2	29	1
8	1	39	1
10	1	Total	21

The total number of circular triads was 242 with a mean of 11·5. Only one boy was entirely consistent. On the other hand, for $n = 13$ the maximum number of circular triads is 91, with a mathematical expectation of 71·5. It is thus clear that, except perhaps for one boy, we cannot suppose that any boy allotted preferences at random. We are again led to conclude that the boys are genuinely capable of making distinctions, and that consistently on the whole. Half the boys have coefficients of consistence ζ greater than 0·92.

We conclude that the boys can make preferences and that in their view the subjects are sufficiently different to enable a reasonably consistent set of preferences to be made. So far as these data are concerned we would see no objection to the assumption that a *scale* of preferences can be set up. With this in mind we can say that the value of u indicates a certain amount of agreement, though not a strong one, between the boys as to which subjects they prefer.

(*b*) 25 girls, 11 school subjects. Table VIII shows the data.

TABLE VIII

Preferences of 25 girls in 11 subjects

	1	2	3	4	5	6	7	8	9	10	11	Total
1. Gymnastics	—	10	19	17	20	17	21	21	21	18	22	186
2. Science	15	—	12	15	17	15	21	19	18	16	17	165
3. Art	6	13	—	16	16	18	10	17	16	19	16	147
4. Domestic Science	8	10	9	—	16	11	13	15	14	11	14	121
5. History	5	8	9	9	—	14	18	12	13	15	18	121
6. Arithmetic	8	10	7	14	11	—	12	13	12	16	18	121
7. Geography	4	4	15	12	7	13	—	14	15	14	14	112
8. English Literature	4	6	8	10	13	12	11	—	14	13	14	105
9. Religion	4	7	9	11	12	13	10	11	—	11	17	105
10. Algebra	7	9	6	14	10	9	11	12	14	—	12	104
11. English Grammar	3	8	9	11	7	7	11	11	8	13	—	88
											Total	1375

We find $\Sigma = 8928$, $u = 0{\cdot}082$.

For the χ^2 significance test, $\chi^2 = 180{\cdot}3$, $\nu = 62{\cdot}4$ and $\sqrt{(2\chi^2)} - \sqrt{(2\nu - 1)} = 7{\cdot}9$, as before a very significant result.

The distribution of circular triads was

No. of triads	Frequency	No. of triads	Frequency
1	2	17	1
2	2	19	1
3	1	22	1
4	1	23	2
6	1	27	1
8	1	32	1
9	1	35	1
11	2	37	1
12	2	38	1
13	1		—
14	1	Total	25

The total number of circular triads is 382 with a mean 15·28. For $n = 11$ the maximum number of circular triads is 55 with an expectation of 41·25. Several of the girls come very close to this, the worst having a coefficient of consistence equal to 0·31.

We are, however, again led to conclude that the preferences were not allotted at random and that most of the girls are capable of exercising a judgment which is on the whole consistent. There is only a very slight agreement in preferences.

Thus the girls are less consistent and less alike in preferences than the boys.

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THE MEAN AND VARIANCE OF χ^2, WHEN USED AS A TEST OF HOMOGENEITY, WHEN EXPECTATIONS ARE SMALL

By J. B. S. HALDANE, F.R.S.

PEARSON's measure of divergence, χ^2, can be used not only as a test of goodness of fit, but as a test of homogeneity when the expectations are unknown before the samples are observed. Consider the $(m \times n)$-fold table:

a_{11}	a_{12}	a_{13}	...	a_{1m}	s_1
a_{21}	a_{22}	a_{23}	...	a_{2m}	s_2
a_{31}	a_{32}	a_{33}	...	a_{3m}	s_3
.....					...
.....					...
a_{n1}	a_{n2}	a_{n3}	...	a_{nm}	s_n
t_1	t_2	t_3	...	t_m	N

Here each a_{ij} represents a number of individuals observed, and $s_i = \sum_j a_{ij}$, $t_j = \sum_i a_{ij}$, $N = \sum_i s_i = \sum_j t_j$. The table may represent n samples of s_1, s_2, ..., s_n individuals, each sample falling into m classes, t_1, t_2, ..., t_m being the grand totals in each class. Or it may represent m samples of $t_1, t_2, \ldots, t_m$ individuals, each falling into n classes, the class totals being s_1, s_2, ..., s_n. In each case we ask what is the probability of so bad a fit if every sample is taken from the same large population.

The expected value of a_{ij} is clearly $\frac{s_i t_j}{N}$, and

$$\chi^2 = \sum_{ij} \frac{\left(a_{ij} - \frac{s_i t_j}{N}\right)^2}{\frac{s_i t_j}{N}}$$

$$= N \sum_{ij} \frac{(a_{ij}^2)}{(s_i t_j)} - N.$$

Fisher (1922) showed that when every s_i and t_j was sufficiently large, χ^2 has the usual distribution, with $(m-1)(n-1)$ degrees of freedom. Thus its mean is $(m-1)(n-1)$, its variance $2(m-1)(n-1)$. Haldane (1937, 1938, 1939) investigated the exact values of the moments of χ^2 when expectations are small, in $(m \times n)$-fold tables with mn or $m(n-1)$ degrees of freedom, but did not attempt the present problem. This had previously been done by Cochran (1936), in the

restricted case when $m = 2$, and every s_i is equal. Cochran used a method based on the use of characteristic functions. My own method is entirely elementary, and completely exact, though rather tedious. My results are very close to Cochran's, but the accurate values are worth putting on record.

THE MEAN VALUE OF χ^2

Given the marginal totals, the probability of any particular distribution is

$$P = \frac{\prod_i (s_i!) \prod_j (t_j!)}{N! \prod_{ij} (a_{ij}!)}.$$

And this is so whatever may be the true expectations. Thus, if in the sample s_i the expectation of the observation a_{ij} is $s_i p_j$, so that that of t_j is Np_j, then the probability of the given distribution is

$$\frac{\prod_j (p_j^{t_j}) \prod_i (s_i!)}{\prod_{ij} (a_{ij}!)}.$$

But the probability of the given marginal totals t_j is

$$\frac{\prod_j (p_j^{t_j}) N!}{\prod_j (t_j!)}.$$

Hence P is the probability of obtaining any particular distribution with the given sample sizes s_i and class totals t_j.

If $E(x)$ denotes the sampling expectation of x, then $E(a_{ij}) = \Sigma P a_{ij}$, summation being taken over all samples possible with the given marginal totals. Hence

$$E(a_{ij}) = \frac{s_i t_j}{N} \Sigma \frac{(s_i - 1)!\,(t_j - 1)! \prod_k (s_k!) \prod_l (t_l!)}{(N-1)!\,(a_{ij} - 1)! \prod_{kl} (a_{kl}!)},$$

where k assumes all positive integral values between 0 and n except i, and l all positive integral values between 0 and m except j. That is to say the quantity summed is the probability of a set of observations similar to those of the table except that s_i and t_j have been diminished by unity. Hence the sum equals unity.

Similarly
$$E[a_{ij}(a_{ij} - 1)] = \frac{s_i(s_i - 1)\,t_j(t_j - 1)}{N(N-1)},$$

$$E(a_{ij}a_{kj}) = \frac{s_i s_k t_j(t_j - 1)}{N(N-1)},$$

$$E(a_{ij}a_{kl}) = \frac{s_i s_k t_j t_l}{N(N-1)},$$

and so on. Hence

$$E(\chi^2) = \overline{\chi^2} = NE\left[\sum_{ij}\frac{(a_{ij}^2)}{(s_i t_j)}\right] - N$$

$$= N\sum_{ij}\left[\frac{1}{s_i t_j}E(a_{ij}^2)\right] - N$$

$$= \sum_{ij}\left[\frac{(s_i-1)(t_j-1)}{N-1}+1\right] - N$$

$$= \frac{1}{N-1}\sum_{ij}(s_i t_j - s_i - t_j + N) - N$$

$$= \frac{1}{N-1}(N^2 - nN - mN + mnN) - N$$

$$= \frac{(m-1)(n-1)N}{N-1}, \qquad \text{.....(1)}$$

a result previously obtained by Bartlett (1937).

This clearly becomes $(m-1)(n-1)$ when N tends to infinity. In the case where $m = 2$, and every $s_i = s$, we have

$$\overline{\chi^2} = \frac{(n-1)ns}{ns-1}. \qquad \text{.....(2)}$$

Cochran gave $$\overline{\chi^2} = n - 1 + \frac{ns - s + 1}{ns^2}.$$

This exceeds (2) by $\dfrac{s-1}{ns^2(ns-1)}$, and is therefore correct to $0(n^{-1})$. It will be noted that the mean depends only on the number of degrees of freedom and the grand total N. The higher moments involve the marginal totals s_i and t_j, and are therefore more complicated. It is clear that the expectation is diminished because, in sampling from a finite group of N individuals, $E(a_{ij})$ has the same value as in sampling from an infinite population with the same frequency, but $E(a_{ij}^2)$ is less.

The variance in a $(2 \times n)$-fold table with equal samples

Owing to the complexity of the general expression for the variance, we shall first calculate it for the $(2 \times n)$-fold table:

a_1	a_2	a_3	...	a_n	A
b_1	b_2	b_3	...	b_n	B
s	s	s	...	s	N

Here n samples of s are taken, and each falls into two classes, say a_i healthy and b_i diseased. The totals are A and B, and $N = ns = A + B$. It will be convenient to put $A = pN$, $B = qN$.

$$\chi^2 = \frac{1}{spq}\Sigma(sp - a_i)^2$$

$$= \frac{1}{spq}\Sigma a_i^2 - \frac{nps}{q},$$

$$\left(\chi^2 + \frac{nps}{q}\right)^2 = \frac{1}{s^2p^2q^2}(\Sigma a_i^2)^2.$$

Now
$$(\Sigma a_i^2)^2 = \sum_i a_i^4 + 2\sum_{ij} a_i^2 a_j^2,$$

summation being taken over all unequal pairs of values of i and j.
Hence

$$E(\Sigma a_i^2)^2 = nE(a_i^4) + n(n-1)E(a_i^2 a_j^2)$$

$$= nE[a_i(a_i-1)(a_i-2)(a_i-3) + 6a_i(a_i-1)(a_i-2) + 7a_i(a_i-1) + a_i]$$
$$+ n(n-1)E[a_i(a_i-1)a_j(a_j-1) + 2a_i(a_i-1)a_j + a_i a_j]$$

$$= \frac{A(A-1)(A-2)(A-3)}{N(N-1)(N-2)(N-3)}[ns(s-1)(s-2)(s-3) + n(n-1)s^2(s-1)^2]$$

$$+ \frac{A(A-1)(A-2)}{N(N-1)(N-2)}[6ns(s-1)(s-2) + 2n(n-1)s^2(s-1)]$$

$$+ \frac{A(A-1)}{N(N-1)}[7ns(s-1) + n(n-1)s^2] + \frac{A}{N}ns.$$

Since $N = ns$, $A = pns$, we have

$$\frac{1}{pns}E(\Sigma a_i^2)^2 = \frac{(A-1)(A-2)(A-3)}{(N-1)(N-2)(N-3)}(s-1)[ns^2 - (n+4)s + 6]$$

$$+ \frac{2(A-1)(A-2)}{(N-1)(N-2)}(s-1)[(n+2)s - 6]$$

$$+ \frac{A-1}{N-1}[(n+6)s - 7] + 1,$$

$$E(\Sigma a_i^2)^2 = \frac{pn^2s^3}{(ns-1)(ns-2)(ns-3)}[p^3n^3s^3(s-1)\{ns^2 - (n+4)s + 6\}$$

$$+ 2p^2n(n-1)s(s-1)(ns-6)$$

$$+ p(n-1)\{n(n+1)s^2 - 2(3n-4)s - 2\} - 2(n-1)(s-1)].$$

So the variance of χ^2 is given by

$$V(\chi^2) = \frac{E(\Sigma a_i^2)^2}{p^2q^2s^2} - \left[\frac{(n-1)\,ns}{(ns-1)} + \frac{pns}{q}\right]^2$$

$$= \frac{E(\Sigma a_i^2)^2}{p^2q^2s^2} - \frac{n^2s^2[pn(s-1)+n-1]^2}{q^2(ns-1)^2}$$

$$= \frac{n^2s}{pq^2(ns-1)^2\,(ns-2)\,(ns-3)}[(ns-1)\{p^3n^2s(s-1)\,(ns^2-\overline{n-4}s+6)$$

$$+2p^2n(n-1)\,s(s-1)\,(ns-6)$$

$$+p(n-1)\,(n\overline{n+1}s^2-6\overline{n-8}s-2)-2(n-1)\,(s-1)\}$$

$$-ps(ns-2)\,(ns-3)\{pn(s-1)+n-1\}^2]$$

$$= \frac{2n^2(n-1)\,s(s-1)}{pq^2(ns-1)^2\,(ns-2)\,(ns-3)}[p^3n^2s^2-2p^2n^2s^2+p(n^2s^2+ns-1)-(ns-1)]$$

$$= \frac{2n^2(n-1)\,s(s-1)\,(pqn^2s^2-ns+1)}{pq(ns-1)^2\,(ns-2)\,(ns-3)}$$

$$= \frac{2n^4(n-1)\,s^3(s-1)}{(ns-1)^2\,(ns-2)\,(ns-3)}\left(1-\frac{ns-1}{AB}\right)$$

$$= \frac{2(n-1)\,N^3(N-n)}{(N-1)^2\,(N-2)\,(N-3)}\left(1-\frac{N-1}{AB}\right). \qquad \text{.....(3)}$$

This becomes $2(n-1)$ when A and B tend to infinity.

When A and B are both of order N, that is to say neither p nor q is small, it becomes

$$2(n-1)\left[1-\left(n+\frac{1}{pq}-7\right)N^{-1}\right]+0(N^{-2}).$$

Cochran's expression is $2(n-1)\,(1-nN^{-1})+0(N^{-2})$, which is accurate when $p=\frac{1}{2}\pm\frac{\sqrt{21}}{14}$, i.e. 0·8273 or 0·1727. It will be seen that the variance is diminished when $n>7-\frac{1}{pq}$, which is certainly the case if $n>3$. If A remains finite when N and B tend to infinity we find $\overline{\chi^2}=n-1$,

$$V(\chi^2)=2(n-1)\left(1-\frac{1}{A}\right),$$

a formula already given by Haldane (1937).

Formula (3) may be compared with the values given by Haldane (1937) for

a $(n \times 2)$-fold table with n degrees of freedom. In the terminology of this paper they are:

$$\overline{\chi^2} = n,$$

$$V(\chi^2) = 2n + \frac{n}{s}\left(\frac{1}{pq} - 6\right).$$

Similarly the limiting case given above may be compared with the values for a n-fold table with n degrees of freedom, namely,

$$\overline{\chi^2} = n,$$

$$V(\chi^2) = 2n + \frac{n}{a}.$$

THE VARIANCE FOR A $(m \times n)$-FOLD TABLE

This of course includes the variance found above as a special case. However, as the summations involved are somewhat hard to follow, the simpler algebra for the special case has been given separately.

$$\chi^2 = N \sum_{ij}\left(\frac{a_{ij}^2}{s_i t_j}\right) - N.$$

Hence $$N^{-2}(\chi^2 + N)^2 = \left[\sum_{ij}\left(\frac{a_{ij}^2}{s_i t_j}\right)\right]^2$$

$$= \sum_{ij}\left(\frac{a_{ij}^4}{s_i t_j^2}\right) + \sum_{ikj}\left(\frac{a_{ij}^2 a_{kj}^2}{s_i s_k t_j^2}\right) + \sum_{ijl}\left(\frac{a_{ij}^2 a_{il}^2}{s_i^2 t_j t_l}\right) + \sum_{ijkl}\left(\frac{a_{ij}^2 a_{kl}^2}{s_i s_j t_j t_l}\right).$$

Here summation over the values of i, k on the one hand, and j, l on the other, are independent and commutative. $\sum_{ik}(x) = \sum_i\left[\sum_k(x)\right]$, where k assumes all values from o to n inclusive, except i. $\sum_{jl}(x)$ has a similar meaning.

Thus $\sum_i(1) = n$, $\sum_i(s_i) = N$, $\sum_i\left[1 + \sum_k(1)\right] = n^2$, $\sum_i\left[s_i + \sum_k s_k\right] = nN$, and so on.

Let $$S = \sum_i s_i^{-1},\ T = \sum_j t_j^{-1}.$$

Then

$$a_{ij}^4 = a_{ij}(a_{ij}-1)(a_{ij}-2)(a_{ij}-3) + 6a_{ij}(a_{ij}-1)(a_{ij}-2) + 7a_{ij}(a_{ij}-1) + a_{ij},$$

$$a_{ij}^2 a_{kj}^2 = a_{ij}(a_{ij}-1)a_{kj}(a_{kj}-1) + a_{ij}(a_{ij}-2)a_{kj} + a_{ij}a_{kj}(a_{kj}-1) + a_{ij}a_{kj},$$

and so on. The expectations of the two middle terms in the last expression are of course equal. Remembering that

$$E[a_{ij}(a_{ij}-1)(a_{ij}-2)(a_{ij}-3)] = \frac{s_i(s_i-1)(s_i-2)(s_i-3)\,t_j(t_j-1)(t_j-2)(t_j-3)}{N(N-1)(N-2)(N-3)},$$

and $$E[a_{ij}(a_{ij}-1)a_{kj}(a_{kj}-1)] = \frac{s_i(s_i-1)s_k(s_k-1)\,t_j(t_j-1)(t_j-2)(t_j-3)}{N(N-1)(N-2)(N-3)},$$

and so on, we have

$$
\begin{aligned}
&N^{-2}E[(\chi^2+N)^2]\\
&=\frac{1}{N(N-1)(N-2)(N-3)}\Big[\sum_{ij}\frac{(s_i-1)(s_i-2)(s_i-3)(t_j-1)(t_j-2)(t_j-3)}{s_it_j}\\
&\quad+\sum_{ikj}\frac{(s_i-1)(s_k-1)(t_j-1)(t_j-2)(t_j-3)}{t_j}\\
&\quad+\sum_{ijl}\frac{(s_i-1)(s_i-2)(s_i-3)(t_j-1)(t_l-1)}{s_i}+\sum_{ikjl}(s_i-1)(s_k-1)(t_j-1)(t_l-1)\Big]\\
&\quad+\frac{1}{N(N-1)(N-2)}\Big[6\sum_{ij}\frac{(s_i-1)(s_i-2)(t_j-1)(t_j-2)}{s_it_j}\\
&\quad+2\sum_{ikj}\frac{(s_i-1)(t_j-1)(t_j-2)}{t_j}+2\sum_{ijl}\frac{(s_i-1)(s_i-2)(t_j-1)}{s_i}\\
&\quad+2\sum_{ikjl}(s_i-1)(t_j-1)\Big]\\
&=\frac{1}{N(N-1)}\Big[7\sum_{ij}\frac{(s_i-1)(t_j-1)}{s_it_j}+\sum_{ikj}\frac{t_j-1}{t_j}+\sum_{ijl}\frac{s_i-1}{s_i}+\sum_{ikjl}(1)\Big]+\frac{1}{N}\sum_{ij}\Big(\frac{1}{s_it_j}\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&N^{-1}E[(\chi^2+N)^2]\\
&=\frac{1}{(N-1)(N-2)(N-3)}\Big[\sum_i(s_i^2-6s_i+11-6s_i^{-1})+\sum_{ik}(s_i-1)(s_k-1)\Big]\\
&\quad\times\Big[\sum_j(t_j^2-6t_j+11-6t_j^{-1})+\sum_{jl}(t_j-1)(t_l-1)\Big]\\
&\quad+\frac{2}{(N-1)(N-2)}\Big[2\sum_i(s_i-3+2s_i^{-1})\sum_j(t_j-3+2t_j^{-1})\\
&\quad+\Big\{\sum_i(s_i-3+2s_i^{-1})+\sum_{ik}(s_i-1)\Big\}\Big\{\sum_j(t_j-3+2t_j^{-1})+\sum_{jl}(t_j-1)\Big\}\Big]\\
&\quad+\frac{1}{N-1}\Big[6\sum_i(1-s_i^{-1})\sum_j(1-t_j^{-1})+\Big\{\sum_i(1-s_i^{-1})+\sum_{ik}(1)\Big\}\\
&\quad\times\Big\{\sum_j(1-t_j^{-1})+\sum_{jl}(1)\Big\}\Big]+\frac{1}{N}\sum_i(s_i^{-1})\sum_j(t_j^{-1})\\
&=\frac{1}{(N-1)(N-2)(N-3)}[N^2-2(n+2)N+n^2+10n-6S]\\
&\quad\times[N^2-2(m+2)N+m^2+10m-6T]+\frac{2}{(N-1)(N-2)}\\
&\quad\times[2(N-3n+2S)(N-3m+2T)+(nN-n^2-2n+2S)(mN-m^2-2m+2T)]\\
&\quad+\frac{1}{N-1}[6(n-S)(m-T)+(n^2-S)(m^2-T)]+ST.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&N^{-1}(N-1)(N-2)(N-3)\,E[(\chi^2+N)^2]\\
&\quad=[N^2-2(n+2)N+n^2+10n-6S][N^2-2(m+2)N+m^2+10m-6T]\\
&\qquad+2(N-3)[(nm+2)N^2\\
&\qquad-\{nm(n+m)+4nm+6(n+m)-2(n+2)T-2(m+2)S\}N\\
&\qquad+n^2m^2+2nm(n+m)+22nm-2(n^2+8n)T-2(m^2+8m)S+12ST]\\
&\qquad+(N-2)(N-3)[n^2m^2+6nm-(n^2+6n)T-(m^2+6m)S+7ST]\\
&\qquad+(N-1)(N-2)(N-3)ST\\
&\quad=N^4+2(\alpha-\beta-2)N^3+[(\alpha-\beta)^2-6\alpha+6\beta+4]N^2\\
&\qquad+(-3\alpha^2+8\alpha\beta-4\beta^2+6\alpha-4\beta)N+\alpha^2-2\alpha\beta+4\alpha-[(n^2+2n-2)T\\
&\qquad+(m^2+2m-2)S]N^2+[(n^2-2n)T+(m^2-2m)S]N+N^2(N+1)ST,
\end{aligned}
$$

where $\alpha=nm$, $\beta=n+m$.

But
$$
\begin{aligned}
E(\chi^2+N)&=\frac{(n-1)(m-1)N}{N-1}+N\\
&=\frac{N(N+\alpha-\beta)}{N-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&N^{-1}(N-1)^2(N-2)(N-3)\,V(\chi^2)\\
&\qquad=(N-1)[N^4+2(\alpha-\beta-2)N^3+\text{etc.}]-N(N-2)(N-3)(N+\alpha-\beta)^2\\
&\qquad=2(\alpha-\beta+1)N^3+(\alpha^2+2\beta-4)N^2-2(\alpha^2-\alpha\beta+\beta^2+\alpha-2\beta)N\\
&\qquad\quad-\alpha(\alpha-2\beta+4)-[(n^2+2n-2)T+(m^2+2m-2)S]N^2(N-1)\\
&\qquad\quad+[(n^2-2n)T+(m^2-2m)S]N(N-1)+STN^2(N^2-1).
\end{aligned}
$$

Thus

$$
\begin{aligned}
V(\chi^2)=\frac{N}{(N-1)(N-2)(N-3)}\\
\times\left[\frac{\begin{array}{r}2(n-1)(m-1)N^3+(n^2m^2+2n+2m-4)N^2\\-2\{nm(n-1)(m-1)+(n+m)(n+m-2)\}N-nm(n-2)(m-2)\end{array}}{N-1}\right.\\
\left.-\{(n^2+2n-2)T+(m^2+2m-2)S\}N^2\right.\\
\left.+\{n(n-2)T+m(m-2)S\}N+STN^2(N+1),\right] \qquad \text{.....(4)}
\end{aligned}
$$

where S and T have been defined on p. 351 above.

When there are n samples of $s_1, s_2, s_3, \ldots, s_n$ members, falling into two classes whose totals are A and B, we have $m = 2$, $T = \dfrac{N}{AB}$, $S = \Sigma s_i^{-1}$, so

$$V(\chi^2) = \frac{N^2}{(N-1)^2(N-2)(N-3)}\left[2(n-1)N^2 + 2n(2n+1)N - 6n^2 - 6N(N-1)\Sigma s_i^{-1} + \frac{N(N-1)}{AB}\{N(N+1)\Sigma s_i^{-1} - (n^2+2n-2)N + n(n-2)\}\right]. \quad \ldots\ldots(5)$$

When every $s_i = s$, this reduces to formula (3). In the case of the fourfold table $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array}$, it becomes:

$$V(\chi^2) = \frac{N^2}{(N-1)(N-2)(N-3)}\left[\frac{2(N^2+10N-12)}{N-1} - 6N^2\left\{\frac{1}{(a+b)(c+d)} + \frac{1}{(a+c)(b+d)}\right\} + \frac{N^3(N+1)}{(a+b)(c+d)(a+c)(b+d)}\right]. \quad \ldots(6)$$

Discussion

The expressions for the higher moments would clearly be a great deal more complicated. The above calculations have, I think, a twofold interest. They show that the loss of one degree of freedom arises from the fact that we are sampling from a finite, and not an infinite, aggregate. And they point the way towards an exact treatment of the problem of curve fitting, for which Pearson originally designed the measure χ^2. In the case of a $(n \times 2)$-fold table with $(n-1)$ degrees of freedom we are, in effect, asking whether the observations give a satisfactory fit to a horizontal straight line $y = k$, where y is the observed frequency of a type within a sample. If we were trying to fit a line $y = k + lx$, we should have $n-2$ degrees of freedom, x having a different value for each sample. If we were trying to fit a normal curve we should have $n-3$ degrees, and so on. Jeffreys (1938) has pointed out the great difficulties of curve-fitting in such cases when expectations are small. It is clear that the expected value of χ^2 in such a case is not exactly $n-3$. It may turn out to be slightly greater. Thus an investigation of the actual law of error in a particular type of observation will demand an extension of the present investigation to cases where several more degrees of freedom are lost.

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Editorial Note. The expression (3) of p. 350 above for the variance of χ^2 in a $(2 \times n)$-fold table corresponds to that given by B. L. Welch in 1938 ("On tests for homogeneity," *Biometrika*, **30**, 158, equation (28)). E. S. P.

A NOTE ON THE STATISTICAL ANALYSIS OF SENTENCE-LENGTH AS A CRITERION OF LITERARY STYLE

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Some years ago I made a number of calculations of the frequency distribution of words of different length in different books to see to what extent authors kept to a definite distribution and so perhaps might be identified by such a method. The results obtained, however, were not striking and the work was put at one side.

Mr Udny Yule (1939), however, has attacked the problem of authorship from the angle of the variation in sentence length, and this appears to be a much more fertile method of approach.

Mr Yule shows that the frequency distribution of sentence length (i.e. number of words between successive full stops) is of the skew type and by comparing in two different manuscripts, the mean, the median, quartiles and deciles he is able to produce convincing mathematical evidence on the identity or otherwise of their authorship.

Mr Yule does not comment on the skew distribution further than to state (p. 371) "they are not of the Poisson type, but of the type in which the square of the standard deviation largely exceeds the mean".

When I converted some of Yule's tables into diagrams I was struck by their general resemblance to certain skew distributions with which I have recently been dealing in some Entomological problems, and which distributions, I found, became normal and symmetrical if the logarithm of the number was taken as a basis for subdivision into groups instead of the number itself (see Williams, 1927).

I was unable to test this transformation on Yule's figures as he unfortunately does not give the original data, but only the word length of sentences in groups of five; so it was necessary to obtain some new data.

These I obtained by counting the number of words in each of 600 sentences from the following three books:

(1) G. K. Chesterton, *A Short History of England*, 1917.

(2) H. G. Wells, *The Work, Wealth and Happiness of Mankind.*

(3) G. Bernard Shaw, *An Intelligent Woman's Guide to Socialism.*

All three works deal with the exposition of somewhat similar sociological subjects and none of them are in the "conversational" style.

The selection of the sentences was randomized as follows. Each of the books is divided up into chapters, sections or both. In Chesterton's book the first 30

sentences were counted in each of the first 20 chapters. In Wells's book the first 10 sentences were counted in each chapter subdivision up to chapter VII, division 11. In Shaw's book the first 15 sentences in each of sections 1–40 were taken. In each case the greater part of the book was covered.

The original data thus obtained are shown diagrammatically in Fig. 1. Each of the distributions is of the typical skew type obtained by Yule: Shaw is the most extreme and varies from 3 words to 143; Wells is less skew and ranges from 3 to 91; while the Chesterton curve is the least skew and varies from 5 to 91 with only two values over 60.

From Table I it will be seen that the arithmetic mean number of words per sentence is 25·87 for Chesterton, 24·11 for Wells and 31·23 for Shaw. The medians are also different and presumably the quartiles and deciles, but these latter were not calculated.

TABLE I

Frequency constants of the distributions of sentence length

	Chesterton	Wells	Shaw
Number of sentences	600	600	600
Number of words	15,521	14,463	18,735
Arithmetic mean no. of words	25·87	24·11	31·23
Median no. of words	25·3	20·8	26·0
Mean log no. of words	1·37	1·31	1·39
Geometrical mean no. of words	23·5	20·5	24·5
Standard deviation of mean log	0·200	0·237	0·290
Standard error of mean log	0·0080	0·0095	0·0112

If, however, instead of taking the frequency distribution of the actual number of words per sentence we take that of the logarithm of the number we get the distributions shown in Figs. 2–4. They undoubtedly show a very close resemblance to the "normal distribution". The mean log and standard deviation for Chesterton is $1{\cdot}37 \pm 0{\cdot}20$; for Wells $1{\cdot}31 \pm 0{\cdot}24$ and for Shaw $1{\cdot}39 \pm 0{\cdot}29$. The standard error of the mean is, owing to the large number of observations, in all cases very small and approximately $\pm 0{\cdot}01$.

On each of the three figures is superimposed a normal curve of the same area, mean and standard deviation and it will be seen how closely it fits the observed values.

The following comments may, however, be made:

(1) The greater irregularity of the observed values in the lower portion of the distribution is due to the irregular distribution of the logarithms of integers when grouped in small artificial divisions as in the present case. Thus there is no logarithm of an integer between 0·01 and 0·25; none between 0·61 and 0·65 and between 0·71 and 0·75. On the other hand, there are two between 1·11

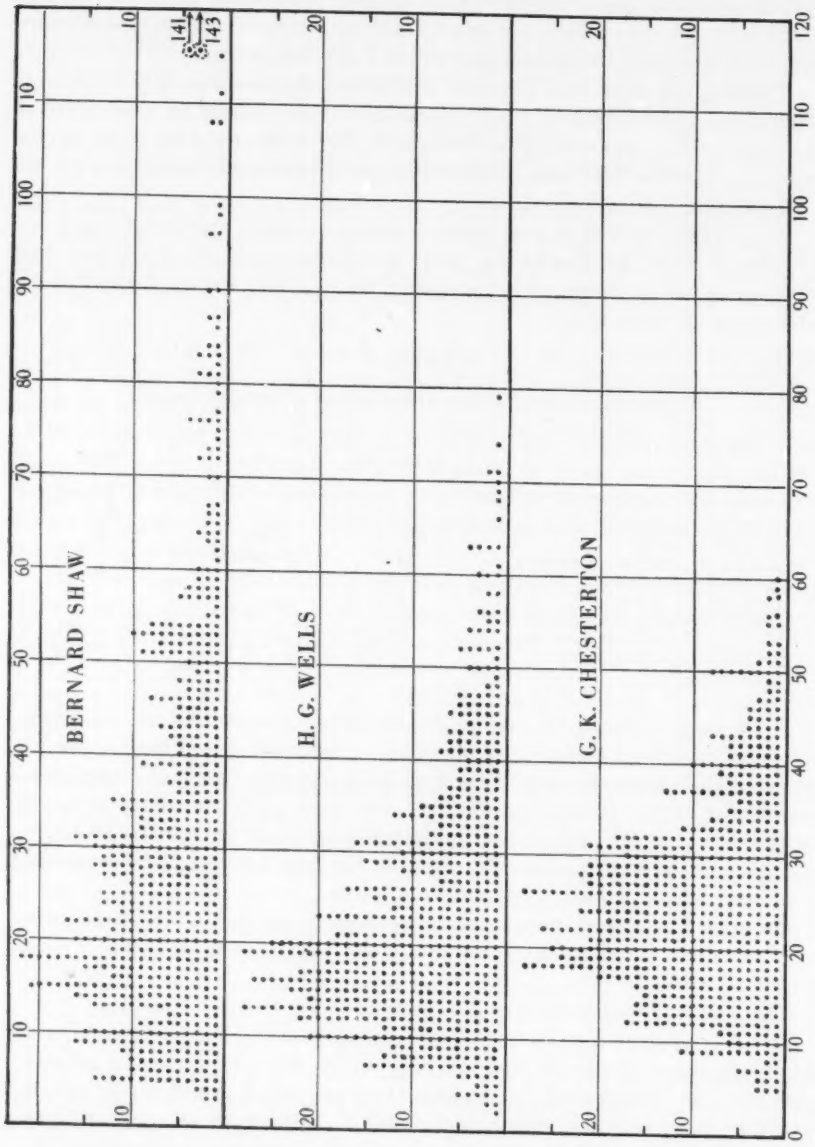
BERNARD SHAW
H. G. WELLS
G. K. CHESTERTON
141
143
0
10
20
30
40
50
60
70
80
90
100
110
120

Fig. 1.

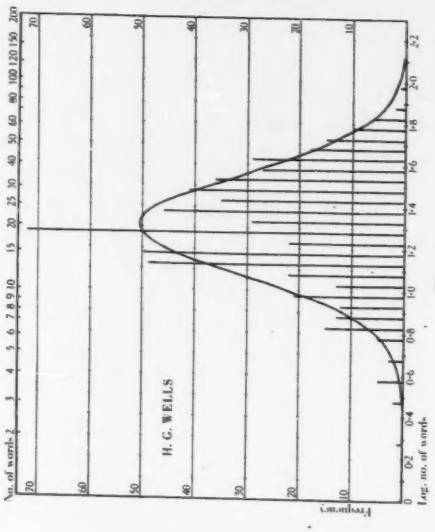

Fig. 3.

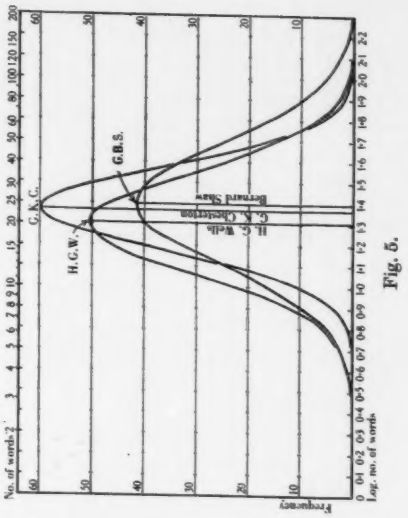

Fig. 5.

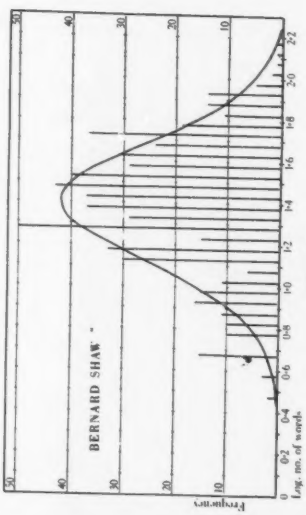

Fig. 2.

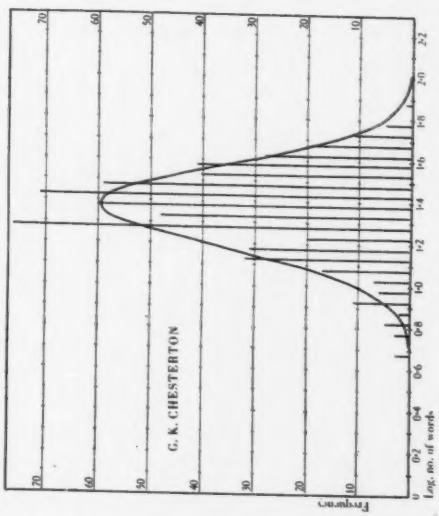

Fig. 4.

and 1·15; two between 1·16 and 1·20; only one between 1·21 and 1·25 and three between 1·26 and 1·30. Thus in all three diagrams the frequency of 1·21–1·25 is well down and that of 1·26–1·30 is far up. Some process of smoothing would undoubtedly eliminate these irregularities, but it was thought better to leave the data in their original form and draw attention to the sources of irregularity.

(2) There appears to be on all three diagrams a slight shortage of high values and a slight excess of low ones. On the latter I have no comment but it appears possible that the small deficit in the longer sentences might easily be due to a biased effect introduced by the habit that many writers have of cutting up unusually long sentences into their component parts when reading over their manuscript or proofs.

On the assumption that the normal curve is a sound representation of the frequency distribution of the log number of words in sentences, Fig. 5 has been prepared which shows the means and normal curves for the three books superimposed on one another. The means are close together but the distributions are very different.

The difference in means between Shaw and Wells is 0·09 and the standard error of the difference is only 0·015. Thus the difference is six times the standard error and hence certainly significant. Between Shaw and Chesterton the difference of the means is barely significant but that between the standard deviations is quite striking.

If the above reasoning is correct, it is unnecessary for the comparison between two documents to compare arithmetic means, medians, quartiles and deciles, but only the log mean and the log standard deviation; all other comparisons are included in these.

It follows also that Mr Bernard Shaw, while undoubtedly under the impression that he was punctuating at his own free will, was for this particular book hide-bound within the limits of

$$Z = \frac{1}{0{\cdot}29\sqrt{(2\pi)}} \exp\left[\frac{(1{\cdot}4-x)^2}{2(0{\cdot}29)^2}\right],$$

while similarly Mr Wells was writing under the restricting influence of

$$Z = \frac{1}{0{\cdot}24\sqrt{(2\pi)}} \exp\left[\frac{(1{\cdot}3-x)^2}{2(0{\cdot}24)^2}\right],$$

where Z is the frequency and x the logarithm of the number of words per sentence.

It is also perhaps worthy of passing comment that the curve representing Mr Shaw is short and broad while that representing Mr Chesterton is tall and slender; which shows how necessary it is to use these curves only for the purpose for which they were originally designed.

Perhaps something might be added on the meaning in words of the above mathematical transformation. If the log distribution is normal we can infer

that the extent to which a sentence in the process of writing is likely to vary is at any level proportional to the length of the sentence. Thus when he is thinking in short sentences of about 10 words an author is likely to vary say from 8 to 12 words; when he is thinking in longer sentences of say 100 words he will vary from 80 to 120. In other words the variations are proportional or geometric and do not merely involve the addition or subtraction of x words at all levels. Further, if the geometric mean is taken as a basis, sentences between this and half its length are as frequent as those between it and twice its length; sentences down to one-quarter its length are as likely to occur as sentences up to four times its length; and so on.

If the arithmetic mean were the true basis then sentences of 10 words more than the arithmetic mean would be as likely to occur as sentences of 10 words less, and it is easy to see that this is not the case.

Before the whole theory of the use of such distributions for separating works of different authorships can be fully accepted it will of course be necessary to study the results obtained from many different works by the same author, in different styles, on different subjects and at different periods of his life. From these it may be possible to find what variation can occur "within authors" as compared with "between authors". This note is not meant to deal with this basic problem but only to draw attention to the simplification of the method of approach to such a problem by the use of a transformation which produces a normal instead of a skew distribution.

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APPLICATIONS OF THE NON-CENTRAL t-DISTRIBUTION

By N. L. JOHNSON and B. L. WELCH

1. Introduction

Statistical problems arising in connexion with the normal distribution are simplified by the circumstance that the sample mean $\bar{x}$ and standard deviation s* are jointly sufficient estimates of the corresponding population parameters ξ and σ. Also the distributions of $\bar{x}$ and s have simple forms and together with Student's t-distribution provide complete solutions of any problems of testing hypotheses or of estimating fiducial limits relating to either ξ or σ singly. In the present paper we shall consider some of the questions which arise when our main concern is not with either ξ or σ alone, but with some function of the two. In particular we shall consider a number of cases which all lead to the use of what has been called the non-central t-distribution. Tables of the probability integral of this distribution will be given in a form suitable for the solution of the problems raised.

2. The non-central t-distribution

Let z be a quantity distributed normally about zero with unit standard deviation and let w be a quantity distributed independently as χ^2/f, where f is the number of degrees of freedom of the χ^2. Then if t is defined by the equation

$$t = \frac{z+\delta}{\sqrt{w}}, \qquad \text{.....(1)}$$

where δ is some constant, then t is distributed in a manner depending only on δ and f. This distribution will be termed a non-central t-distribution. When δ equals zero we have the familiar Student's t.

In general the elementary probability distribution of t is given by

$$p(t) = \frac{1}{2^{\frac{1}{2}(f-1)}\Gamma(\frac{1}{2}f)\sqrt{\pi}} e^{-\frac{1}{2}\frac{f\delta^2}{f+t^2}} \left(\frac{f}{f+t^2}\right)^{\frac{1}{2}(f+1)} \int_0^\infty v^f e^{-\frac{1}{2}\left(v-\frac{t\delta}{\sqrt{(f+t^2)}}\right)^2} dv$$

$$= \frac{f!}{2^{\frac{1}{2}(f-1)}\Gamma(\frac{1}{2}f)\sqrt{\pi}} e^{-\frac{1}{2}\frac{f\delta^2}{f+t^2}} \left(\frac{f}{f+t^2}\right)^{\frac{1}{2}(f+1)} Hh_f\left(-\frac{t\delta}{\sqrt{(f+t^2)}}\right), \qquad \text{.....(2)}$$

where

$$Hh_f(x) = \int_0^\infty \frac{v^f}{f!} e^{\frac{1}{2}(v+x)^2} dv. \qquad \text{.....(3)}$$

An account of some of the properties of the above-defined Hh function has been given by R. A. Fisher (1931). He derives a result equivalent to the above equation (2) although with a different notation, his t being equivalent to our $t/\sqrt{f}$ and his

* $s^2 = \Sigma(x-\bar{x})^2/(n-1)$, where n is the sample size.

τ being our $\delta/\sqrt{(f+1)}$. Tables of the Hh function have been calculated by J. R. Airey (1931). These tables, however, are not useful for solving the problems which are considered in the present paper. For these problems the probability integral of the non-central t-distribution is required. Existent tables of this integral (J. Neyman, 1935 and J. Neyman & B. Tokarska, 1936) have been calculated only for one rather restricted purpose. In the present paper much more extensive tables will be provided which, it is hoped, will cover all the applications of the non-central t-distribution likely to be encountered.

The probability integral of non-central t demands a table of triple entry, since the probability that t exceeds t_0, say, depends on f, δ, and t_0. The notations

$$P(f, \delta, t_0) = P(t > t_0 \,|\, f, \delta) \qquad \ldots\ldots(4)$$

will be used to denote this probability.

Often it is necessary to find what value of t_0 is such that the probability (4) will take a specified value ϵ, say. This t_0 will be a function of f, δ, and ϵ and it will be convenient to denote it by $t(f, \delta, \epsilon)$. Thus

$$P\{t > t(f, \delta, \epsilon) \,|\, f, \delta\} = \epsilon.$$

Again, often t_0 will be given and the value of δ which makes (4) take the value ϵ, will be required. It will be convenient to denote this value of δ by $\delta(f, t_0, \epsilon)$. Thus

$$P\{t > t_0 \,|\, f, \delta(f, t_0, \epsilon)\} = \epsilon.$$

Space does not permit the tabling of all the three functions $P(f, \delta, t_0)$, $t(f, \delta, \epsilon)$ and $\delta(f, t_0, \epsilon)$. For reasons which will become clear later in the paper it was decided to table $\delta(f, t_0, \epsilon)$ most fully. Table IV, given at the end of the paper, facilitates the direct calculation of $\delta(f, t_0, \epsilon)$ for seventeen probability levels ϵ. It can also be used without difficulty for calculating $t(f, \delta, \epsilon)$. Table V is an additional short table from which $t(f, \delta, \epsilon)$ can be calculated rather more directly but only for the probability levels $\epsilon = 0{\cdot}05$ and $\epsilon = 0{\cdot}95$. These tables are given in the most suitable form which we have been able to evolve consistent with necessary limitations on size.

Before the tables are described, the next three sections will be devoted to a description of some of the situations where they are required.

3. Coefficient of variation

The first function of ξ and σ to be discussed will be the coefficient of variation $V = \sigma/\xi$. In the practical situations where this index is an appropriate measure of variability, the variable x is usually necessarily positive. Now for a normal population the ratio of the mean to the standard deviation has to be of the order of 3 or more, for the chance of a negative x to be negligible. Strictly speaking, therefore, the sampled population should not be assumed normal if the coefficient of variation is too great. The figure 33 % has often been stated as the permissible

upper limit. In practice coefficients of variation are usually much smaller than this, and it is assumed in the present section that we are dealing with such cases.

An estimate of V is provided by the sample coefficient of variation $v = s/\bar{x}$. Now since we may write

$$\frac{\sqrt{n}}{v} = \frac{\sqrt{n}\,\bar{x}}{s} = \left\{\frac{\sqrt{n}(\bar{x}-\xi)}{\sigma} + \frac{\sqrt{n}\,\xi}{\sigma}\right\} \div \frac{s}{\sigma} \qquad \text{.....(5)}$$

it appears from comparison with (1) that $\sqrt{n}/v$ is distributed as non-central t with $f = (n-1)$ and $\delta = \sqrt{n}/V$ (McKay, 1932). The solution of problems relating to V is therefore easily effected.

For instance, suppose it is desired to test whether the sample contradicts the hypothesis that $V \leqslant V_0$, and that we decide to reject the hypothesis when $v > v_0$, where v_0 is chosen so that $P(v > v_0 \mid V = V_0)$ is equal to some specified small chance ϵ. In the notation of the previous section v_0 will then be given by

$$\frac{\sqrt{n}}{v_0} = t\left(\overline{n-1}, \frac{\sqrt{n}}{V_0}, \overline{1-\epsilon}\right). \qquad \text{.....(6)}$$

Again, consider an example where it is decided to reject a sample as unsatisfactory when v is greater than a given value v_0, and where it is required to know how low the true coefficient of variability should be kept to ensure that the probability of rejection will not exceed a given ϵ, i.e. we require V_0 such that $P(v > v_0 \mid V = V_0) = \epsilon$. In the notation of Section 2, V_0 is given by

$$\frac{\sqrt{n}}{V_0} = \delta\left(\overline{n-1}, \frac{\sqrt{n}}{v_0}, \overline{1-\epsilon}\right). \qquad \text{.....(7)}$$

Or finally suppose that a value of v is observed and an upper fiducial limit of V is required so that the chance is ϵ of this limit being exceeded, i.e. a lower fiducial limit of $\sqrt{n}/V$ is required. Now since

$$P\left\{\frac{\sqrt{n}}{v} > t\left(\overline{n-1}, \frac{\sqrt{n}}{V}, \epsilon\right)\right\} = \epsilon$$

and since the inequality $\qquad \dfrac{\sqrt{n}}{v} > t\left(\overline{n-1}, \dfrac{\sqrt{n}}{V}, \epsilon\right)$

is equivalent to the inequality*

$$\frac{\sqrt{n}}{V} < \delta\left(\overline{n-1}, \frac{\sqrt{n}}{v}, \epsilon\right),$$

it is seen that the required upper fiducial limit of V is

$$\sqrt{n} \Big/ \delta\left(\overline{n-1}, \frac{\sqrt{n}}{v}, \epsilon\right). \qquad \text{.....(8)}$$

* This follows from the fact that $t(f, \delta, \epsilon)$ is a monotonically increasing function of δ.

4. THE POWER OF STUDENT'S t-TEST

Suppose it is desired to test the hypothesis H_0 that the mean ξ of a normal population has the value ξ_0. Student's t-test consists in calculating from the data the quantity

$$t = \frac{\sqrt{n}\,(\bar{x}-\xi_0)}{s} \qquad \text{.....(9)}$$

and referring it to the usual central t-distribution with $f = (n-1)$ degrees of freedom. Thus if the only alternative hypotheses to be taken into consideration are those for which $\xi > \xi_0$, the test will consist in rejecting the hypothesis when $t > t_0$, where t_0 is such that $P(t > t_0 \mid H_0) = \epsilon$, and ϵ is the conventional level of significance. In our notation

$$t_0 = t(\overline{n-1},\ 0,\ \epsilon). \qquad \text{.....(10)}$$

Now when H_0 is not true, but ξ has some alternative value ξ_1, it is often required to know how powerful Student's test will be, i.e. what chance the test will have of rejecting H_0. But we can write (9) in the form

$$t = \left\{\frac{\sqrt{n}\,(\bar{x}-\xi_1)}{\sigma} + \frac{\sqrt{n}\,(\xi_1-\xi_0)}{\sigma}\right\} \div \frac{s}{\sigma} \qquad \text{.....(11)}$$

whence, comparing with (1), it now appears that the quantity calculated from the data is distributed in the non-central t-distribution with $f = (n-1)$ and

$$\delta = \sqrt{n}\,(\xi_1-\xi_0)/\sigma.$$

The power of the test is therefore $P(\overline{n-1}, \sqrt{n}\,(\xi_1-\xi_0)/\sigma, t_0)$ and the value of ξ_1 for which the power reaches any specified value η, say, is given by

$$\frac{\sqrt{n}\,(\xi_1-\xi_0)}{\sigma} = \delta(\overline{n-1}, t_0, \eta). \qquad \text{.....(12)}$$

Tables for evaluating the power of the t-test together with a discussion of their use have been given by J. Neyman (1935) and J. Neyman & B. Tokarska (1936). They are not restricted to the simple case of testing whether a mean has a specified value but apply to all cases in which the t-test is used.

5. PROPORTION DEFECTIVE PROBLEMS

Another class of problem where the non-central t-distribution has an application, occurs when objects are classified as defective or non-defective according to whether they have values of a characteristic exceeding or falling short of a fixed standard. Thus, if an object is defective when the character x exceeds a fixed given level L, then information will often be required about the proportion P falling beyond L in the population. If the population be normal P clearly depends only on the ratio

$$U = \frac{(L-\xi)}{\sigma}.$$

An estimate of U is provided from the sample by calculating

$$u = \frac{(L-\bar{x})}{s}. \qquad \text{.....(13)}$$

Corresponding to deviate u the equivalent normal probability p will then give an estimate of P. This situation has been discussed recently by W. J. Jennett & B. L. Welch (1939).* The transference from U to P and from u to p does of course require that the sampled distribution is really approximately normal and for this reason care must be taken in going out to the "tails" of the distribution (i.e. when U is large and P small).

In order to allow for the sampling errors arising from this method of estimating P we have only to note that (13) may be written

$$\sqrt{n}\, u = \left\{\frac{\sqrt{n}\,(L-\xi)}{\sigma} - \frac{\sqrt{n}\,(\bar{x}-\xi)}{\sigma}\right\} \div \frac{s}{\sigma}. \qquad \text{.....(14)}$$

Hence $\sqrt{n}\,u$ is distributed as non-central t with $f = (n-1)$ and $\delta = \sqrt{n}\,U$. Thus if the proportion P (and hence U) were known the value u_0 of u such that $P(u > u_0) = \epsilon$ would be given by

$$\sqrt{n}\, u_0 = t(\overline{n-1}, \sqrt{n}\, U, \epsilon). \qquad \text{.....(15)}$$

Conversely, given u from a sample, a lower fiducial limit of U is obtained by noting that the inequality

$$\sqrt{n}\, u > t(\overline{n-1}, \sqrt{n}\, U, \epsilon)$$

is equivalent to the inequality

$$\sqrt{n}\, U < \delta(\overline{n-1}, \sqrt{n}\, u, \epsilon). \qquad \text{.....(16)}$$

Hence a lower fiducial limit for U will be given by $\delta(\overline{n-1}, \sqrt{n}\,u, \epsilon)/\sqrt{n}$. In the above analysis the proportion falling beyond L has been termed a "proportion defective". In industrial problems this is often a fair description. More generally the analysis refers to any problem where we are primarily interested in how a measurable object is related to a fixed level. A slightly different problem occurs when we are not given L but are asked to estimate the value of L so that P will have an assigned value. For instance, the value of L may be required so that the chance is 1 in 20 of it being exceeded.

Now if U is the normal deviate which is exceeded with probability P, then the required L is related to ξ and σ by the relation

$$L = \xi + U\sigma.$$

An estimate of L is therefore given by

$$l = \bar{x} + Us. \qquad \text{.....(17)}$$

* For a general discussion of similar problems arising in the control of industrial products the reader may be referred to the British Standards Institution Publication, No. 600 (E. S. Pearson, 1935).

If fiducial limits are required for L they may be obtained by returning to equations (13) and (15). L is not now given, but U is definitely known and therefore (15) provides the value of u_0 such that $P(u > u_0) = \epsilon$. But the inequality

$$u = \frac{L - \bar{x}}{s} > u_0$$

can be reversed to read $$L > \bar{x} + u_0 s.$$

Hence, if we write $$l_\epsilon = \bar{x} + \frac{st(\overline{n-1}, \sqrt{n}\, U, \epsilon)}{\sqrt{n}} \qquad \text{.....(18)}$$

we shall have $P(L > l_\epsilon) = \epsilon$. Hence l_ϵ will be a fiducial limit for L, which will be exceeded by L in a proportion ϵ of cases.

6. Large sample results

Before going on to discuss the distribution of non-central t in general and the method of applying the tables given below, it will be convenient to consider the situation when f is large. The distribution then approaches the normal form. The important question is how quickly.

The first three moments of t are given exactly by the following expressions:

$$\mu_1' = \sqrt{\frac{f}{2}} \frac{\Gamma\left(\frac{f}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{f}{2}\right)} \delta, \qquad \text{.....(19)}$$

$$\mu_2 = \frac{f(1+\delta^2)}{(f-2)} - \mu_1'^2, \qquad \text{.....(20)}$$

$$\mu_3 = \mu_1' \left[\frac{f(2f - 3 + \delta^2)}{(f-2)(f-3)} - 2\mu_2 \right]. \qquad \text{.....(21)}$$

If the gamma functions are expanded in powers of $1/f$ and only the leading terms are retained, these give *approximately*

$$\mu_1' = \delta; \quad \mu_2 = \left(1 + \frac{\delta^2}{2f}\right); \quad \mu_3 = \frac{\delta}{f}\left[3 + \frac{5\delta^2}{4f}\right]; \quad \sqrt{\beta_1} = \frac{\delta\left[3 + \frac{5\delta^2}{4f}\right]}{f\left[1 + \frac{\delta^2}{2f}\right]^{\frac{3}{2}}}. \qquad \text{.....(22)*}$$

In large samples, therefore, t becomes normally distributed about δ with standard deviation $\left(1 + \frac{\delta^2}{2f}\right)^{\frac{1}{2}}$. The rapidity of the approach to symmetry is indicated by the true values of $\sqrt{\beta_1}$ in Table I. For given f the greatest value $\sqrt{\beta_1}$ can take is shown in the last column.

* The term δ^2/f is retained because in most problems δ will be of the order $\sqrt{f}$.

TABLE I

$\sqrt{\beta_1}$ of t for different f and δ

f \ $\delta/\sqrt{f}$	0	1	2	3	∞
4	0·00	3·70	4·61	4·86	5·09
6	0·00	1·67	2·13	2·26	2·38
8	0·00	1·20	1·55	1·65	1·74
12	0·00	0·84	1·10	1·18	1·25
24	0·00	0·53	0·69	0·74	0·79
∞	0·00	0·00	0·00	0·00	0·00

It is seen from Table I that values of $\sqrt{\beta_1}$ of the order of unity are likely to occur fairly frequently in the type of problem which has been discussed in the previous sections. This represents a considerable degree of skewness and another method of making normal approximations, outlined below, is preferable. Before describing this we may note that if t is referred to a normal scale with mean δ and standard deviation $\left(1+\frac{\delta^2}{2f}\right)^{\frac{1}{2}}$ then the following approximations to the quantities defined in section 2 will be obtained:

$$t(f, \delta, \epsilon) = \delta + K_\epsilon\left(1+\frac{\delta^2}{2f}\right)^{\frac{1}{2}}, \qquad \text{.....(23)}$$

where K_ϵ is the deviate of the unit normal curve exceeded with a probability ϵ. Also $\delta(f, t_0, \epsilon)$ will be given by the solution of the equation in δ

$$t_0 = \delta + K_\epsilon\left(1+\frac{\delta^2}{2f}\right)^{\frac{1}{2}}. \qquad \text{.....(24)}$$

On rationalization this equation is quadratic in δ and the correct root to take will be obvious.

Better approximations, which are, however, still based on the normal curve may be obtained as follows. The probability that t exceeds a given value t_0 is by (1) the probability of the inequality

$$t = \frac{(z+\delta)}{\sqrt{w}} > t_0, \qquad \text{.....(25)}$$

and this inequality is equivalent to

$$(-z + t_0\sqrt{w}) < \delta. \qquad \text{.....(26)}$$

Now $(-z)$ is a unit normal deviate, and $\sqrt{w}$, being of the form $\chi/\sqrt{f}$ is very nearly normally distributed even for small f. Since z and $\sqrt{w}$ are independent, $(-z+t_0\sqrt{w})$ must therefore be more nearly normally distributed than $\sqrt{w}$, whatever t_0.*

* This is practically obvious, but a demonstration is given in the Appendix, equation (43).

Hence an upper limit to the skewness of $(-z+t_0\sqrt{w})$ is given for different f by the values of $\sqrt{\beta_1}$ in Table II. Comparison with Table I shows that it is better to take $(-z+t_0\sqrt{w})$ as normally distributed rather than t. The procedure is then as follows:

Write $$\text{mean}\sqrt{w} = a; \quad \sigma_{\sqrt{w}} = \frac{b}{\sqrt{(2f)}}. \qquad \text{.....(27)}$$

Then $(-z+t_0\sqrt{w})$ is taken to be normally distributed with mean at_0 and standard deviation $\left(1+\frac{b^2t_0^2}{2f}\right)^{\frac{1}{2}}$. For given t_0 the value of δ such that $P(t>t_0)$ equals ϵ is given by seeking δ such that the probability is ϵ of (26) being true. This gives

$$\delta(f,t_0,\epsilon) = at_0 - K_\epsilon\left(1+\frac{b^2t_0^2}{2f}\right)^{\frac{1}{2}}. \qquad \text{.....(28)}$$

TABLE II

$\sqrt{\beta_1}$ of $\sqrt{w}$ for different f

f	4	6	8	12	24	∞
$\sqrt{\beta_1}$	0·41	0·32	0·27	0·21	0·15	0·00

Conversely for given δ the value $t(f,\delta,\epsilon)$ which will be exceeded with probability ϵ, will be approximated by solving for t the equation

$$\delta = at - K_\epsilon\left(1+\frac{b^2t^2}{2f}\right)^{\frac{1}{2}}. \qquad \text{.....(29)}$$

On rationalization this becomes a quadratic in t. This method of approximation was given by W. J. Jennett & B. L. Welch (1939) together with a short table of a and b. However, since a and b differ from unity by a quantity of the order $1/f$ and since errors at least of this magnitude are involved in the assumption of normality, it would perhaps have been as logical to take a and b equal to unity in the approximation. The numerical comparisons set out in Table III indicate that nothing is lost by doing this and also show how inferior is the method of approximation which assumes t itself to be normally distributed.

Therefore the approximation which we shall adopt as being the best one, requiring only the use of the normal probability scale, will consist in assuming $(-z+t_0\sqrt{w})$ to be normally distributed about t_0 with standard deviation

$$\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{2}}.$$

The value of δ such that $p(t>t_0)$ equals ϵ will then be approximated by

$$\delta(f,t_0,\epsilon) = t_0 - K_\epsilon\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{2}}. \qquad \text{.....(30)}$$

TABLE III

Values of $\delta/\sqrt{f}$

f	ϵ	$\frac{t_0}{\sqrt{f}}$	$\delta/\sqrt{f}$			
			True	Approx. 1	Approx. 2	Approx. 3
4	0·99	2	0·056	0·736	−0·088	−0·015
4	0·01	2	3·972	12·023	3·848	4·015
4	0·99	5	1·120	2·668	0·565	0·726
4	0·01	5	9·247	29·230	8·840	9·274
24	0·99	2	1·181	1·345	0·956	1·178
24	0·01	2	2·819	3·163	2·800	2·822
24	0·99	5	3·290	3·677	3·213	3·255
24	0·01	5	6·752	7·593	6·687	6·745

The values correspond to certain f, ϵ and $t_0/\sqrt{f}$ and have the property that $p(t>t_0 \mid f, \delta) = \epsilon$. Approximation 1 is the value obtained from equation (23), approximation 2 by using (28) with the correct a and b and approximation 3 by using (30), i.e. taking $a = b = 1$.

Conversely, for given δ, the value $t(f, \delta, \epsilon)$ which will be exceeded with probability ϵ will be given implicitly by

$$\delta = t - K_\epsilon\left(1+\frac{t^2}{2f}\right)^{\frac{1}{2}}, \qquad \text{.....(31)}$$

which on rationalizing and solving becomes

$$t(f, \delta, \epsilon) = \frac{\delta + K_\epsilon\left(1+\frac{\delta^2}{2f}-\frac{K_\epsilon^2}{2f}\right)^{\frac{1}{2}}}{\left(1-\frac{K_\epsilon^2}{2f}\right)}. \qquad \text{.....(32)}$$

The approximations (30) and (32) are useful even in moderate-sized samples. It is not intended, however, to enter here into any detailed discussion of how good the various approximations are. They have been considered primarily because they form the basis of the exact tables which are given at the end of the paper and which are described in the next section.

7. Description and use of tables

The tables are intended to facilitate the exact calculation of the quantities $\delta(f, t_0, \epsilon)$ and $t(f, \delta, \epsilon)$ defined in Section 2. The form in which they are presented was determined largely by the fact that the space they had to occupy was limited, and also to some extent by the method which was adopted to calculate them.

Table IV may be used to give $\delta(f, t_0, \epsilon)$ for 17 probability levels, viz.

ϵ = 0·005, 0·01, 0·025, 0·05, 0·1, 0·2, 0·3, 0·4, 0·5, 0·6, 0·7,
0·8, 0·9, 0·95, 0·975, 0·99 and 0·995.

The basis of the table is the large sample approximation described in the last section, equation (30). The K_ϵ in this equation is simply the normal multiple which is exceeded with probability ϵ. In small samples an error will be committed in taking this form of approximation, and in Table IV we give instead a multiple $\lambda(f, t_0, \epsilon)$, which is such that *exactly*

$$\delta(f, t_0, \epsilon) = t_0 - \lambda(f, t_0, \epsilon)\left(1 + \frac{t_0^2}{2f}\right)^{\frac{1}{2}}. \qquad \text{.....(33)}$$

Even for samples giving f as small as 4, it is found that λ never differs much from K_ϵ. This stability makes interpolation easy so that, for given ϵ, it is possible to provide a table of small compass covering all values of t_0 from $-\infty$ to ∞.

For given ϵ the tabled quantity λ is a function of f and t_0 and accordingly Table IV, for given ϵ, is one of double entry.

Values are given corresponding to $f = 4, 5, 6, 7, 8, 9, 16, 36, 144$ and ∞. The reason for choosing these values of f is that λ differs from K_ϵ by a quantity of the order $1/\sqrt{f}$. The values $f = 9, 16, 36, 144$ and ∞ are such that $12/\sqrt{f} = 4, 3, 2, 1$ and 0 respectively. Hence interpolations for intermediate f's may be simply effected by considering λ as a function of $12/\sqrt{f}$, since we are then dealing with a function tabled at equal intervals.

The choice of the values of t_0 in the table is also determined by the necessity for interpolation to be simple and yet the table not to be unduly large. The whole range of t_0 from $-\infty$ to ∞ has to be covered. This has been done as follows. For $t_0/\sqrt{2f}$ between $-\infty$ and $-0{\cdot}75$, λ is given against the quantity

$$y = \left(1 + \frac{t_0^2}{2f}\right)^{-\frac{1}{2}}. \qquad \text{.....(34)}$$

For $t_0/\sqrt{2f}$ between $-0{\cdot}75$ and $0{\cdot}75$, λ is tabled against

$$y' = \frac{t_0}{\sqrt{2f}}\left(1 + \frac{t_0^2}{2f}\right)^{-\frac{1}{2}} \qquad \text{.....(35)}$$

For $t_0/\sqrt{2f}$ between $0{\cdot}75$ and ∞, λ is again tabled against y. The argument interval for both y and y' is $0{\cdot}1$. It will be noted that $y = (1 - y'^2)^{\frac{1}{2}}$.

From the point of view of interpolation other, perhaps simpler, functions of t_0 could have been used in constructing the table. The choice of y was made because the quantity $\left(1 + \frac{t_0^2}{2f}\right)^{\frac{1}{2}}$ is wanted in any case for substitution into equation (33) after λ has been obtained. y' had to be used in addition because of interpolation difficulties in the middle of the table.

It will be noted that in Table IV only the double entry tables for $\epsilon = 0{\cdot}005, 0{\cdot}01, 0{\cdot}025, 0{\cdot}05, 0{\cdot}1, 0{\cdot}2, 0{\cdot}3, 0{\cdot}4$ and $0{\cdot}5$ are given. For $\epsilon > 0{\cdot}5$ we can use the fact that

$$\delta(f, t_0, \epsilon) = -\delta(f, -t_0, \overline{1-\epsilon}), \qquad \text{.....(36)}$$

a relation which is apparent from the form of equation (1).

To summarize, the steps necessary to find the value of δ such that $P(t > t_0) = \epsilon$ are as follows.

(i) Find $y = \left(1 + \frac{t_0^2}{2f}\right)^{-\frac{1}{2}}$ or $y' = \frac{t_0}{\sqrt{2f}}\left(1 + \frac{t_0^2}{2f}\right)^{-\frac{1}{2}}$ according as $\left|\frac{t_0}{\sqrt{2f}}\right|$ is greater than or less than 0·75.

(ii) If $f > 9$ calculate $12/\sqrt{f}$.

(iii) If ϵ is one of the values 0·005, ..., 0·5, enter the appropriate part of Table IV and obtain $\lambda(f, t_0, \epsilon)$, by interpolating with respect to the quantities obtained in (i) and (ii).

(iv) Calculate $$\delta(f, t_0, \epsilon) = t_0 - \lambda\left(1 + \frac{t_0^2}{2f}\right)^{\frac{1}{2}}.$$

(v) If ϵ is one of the values 0·6, 0·7, ..., 0·995 calculate $\delta(f, -t_0, \overline{1-\epsilon})$ and then change its sign.

Inverse use of Table IV

Next consider the calculation of $t(f, \delta, \epsilon)$, i.e. the situation where δ is given and t_0 is required so that $P(t > t_0) = \epsilon$. In the previous section it was shown that a first approximation will be given by equation (32). The true relation between $t(f, \delta, \epsilon)$ and δ will, however, be obtained by replacing K_ϵ in (32) by the λ which is tabled in Table IV; thus

$$t(f, \delta, \epsilon) = \frac{\delta + \lambda\left(1 + \frac{\delta^2}{2f} - \frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1 - \frac{\lambda^2}{2f}\right)}. \qquad \text{.....(37)}$$

The drawback about this equation is that λ is given in Table IV as a function of t_0, and t_0 is not now known. An iteration method is therefore necessary.

The successive steps in this iteration may be summarized as follows:

(i) The first approximation t_1 is given by (32); thus

$$t_1 = \frac{\delta + K_\epsilon\left(1 + \frac{\delta^2}{2f} - \frac{K_\epsilon^2}{2f}\right)^{\frac{1}{2}}}{\left(1 - \frac{K_\epsilon^2}{2f}\right)}.$$

(ii) Next use Table IV to calculate directly

$$\lambda_1 = \lambda(f, t_1, \epsilon).$$

(iii) A second approximation t_2 is then given by substituting this λ_1 in (37); thus

$$t_2 = \frac{\delta + \lambda_1\left(1 + \frac{\delta^2}{2f} - \frac{\lambda_1^2}{2f}\right)^{\frac{1}{2}}}{\left(1 - \frac{\lambda_1^2}{2f}\right)}.$$

(iv) Then by finding $\lambda_2 = \lambda(f, t_2, \epsilon)$ and substituting in (37) a third approximation t_3 will be obtained. These steps must be repeated until two successive approximations give the same value. In practice this is found to occur very quickly so that there is likely to be no need of more than three approximations.

(v) If ϵ is one of the values 0·6, 0·7, ..., 0·995 calculate $t(f, -\delta, \overline{1-\epsilon})$ and then change its sign.

Use of Table V

The iteration process described above is not very lengthy in actual practice, but even this amount of trouble would be unnecessary if λ were tabled as a function of δ as well as a function of t. Such a table, similar in extent to Table IV, could of course be calculated, but it did not seem worth while to do this, since the inverse method of using Table IV is not difficult. However, it has been thought useful to give for the single probability level $\epsilon = 0{\cdot}05$, a table which can be entered with δ, since very often this conventional 1 in 20 chance is taken as a point of reference in statistical problems.

In Table V, λ is tabled against

$$\eta' = \frac{\delta}{\sqrt{(2f)}}\left(1+\frac{\delta^2}{2f}\right)^{-\frac{1}{4}} \qquad \text{.....(38)}$$

at intervals of 0·1 from -1 to 1. When δ is given, this function of δ may be quickly calculated and then λ may be obtained by interpolating in Table V. The substitution of this value of λ in (37) gives the required $t(f, \delta, 0{\cdot}05)$. As before $t(f, \delta, 0{\cdot}95)$ may also be calculated from the same table, being equal to *minus* $t(f, -\delta, 0{\cdot}05)$.

8. EXAMPLES

In the present section some numerical examples will be worked to illustrate the application of Tables IV and V. Reference will be made to theoretical results obtained in Sections 3–5.

Example 1. In a sample of $n = 25$ a coefficient of variation $v = 2{\cdot}6$ is observed. Obtain from this a "median" estimate of the population coefficient V. By a "median" estimate is meant one for which the chance of it exceeding the true V is 0·50. In other words our estimate is the 50 % fiducial limit. Hence from (8) the required estimate is

$$v_m = \sqrt{25}/\delta(f, t_0, 0{\cdot}5), \qquad \text{.....(39)}$$

where $$t_0 = \frac{\sqrt{25}}{2{\cdot}6} = 1{\cdot}9231 \text{ and } f = 24.$$

We have $$\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{4}} = 1{\cdot}0379; \quad y' = \frac{t_0}{\sqrt{(2f)}}\left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{4}} = 0{\cdot}2674,$$

$$12/\sqrt{f} = 2{\cdot}4495.$$

From Table IV, for $\epsilon = 0{\cdot}5$, we obtain by linear interpolation* with respect to y' and $12/\sqrt{f}$, $\lambda = 0{\cdot}0197$. Hence from (33)

$$\delta = 1{\cdot}9231 - (0{\cdot}0197)(1{\cdot}0379) = 1{\cdot}9027.$$

Therefore from (39), $v_m = 2{\cdot}628$.

Example 2. A sample of $n = 10$ measurements of a normally distributed character x is given. The mean $\bar{x}$ is 4·7 and the standard deviation s is 0·2. Obtain 10 % fiducial limits for the proportion P in the population exceeding $x = 5{\cdot}0$.

As is shown in Section 5 this is equivalent to finding limits for $U = (5{\cdot}0 - \xi)/\sigma$. We calculate first $u = (5{\cdot}0 - \bar{x})/s = 1{\cdot}5$. Substituting this into (16) the upper and lower 10 % points for U are given by $\delta(9, 4{\cdot}743, 0{\cdot}9)/\sqrt{10}$ and $\delta(9, 4{\cdot}743, 0{\cdot}1)/\sqrt{10}$.

We have $$\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{2}} = 1{\cdot}5; \quad y = \left(1+\frac{t_0^2}{2f}\right)^{-1} = 0{\cdot}6667.$$

From Table IV for $\epsilon = 0{\cdot}1$ we obtain by linear interpolation*

$$\lambda(9, 4{\cdot}743, 0{\cdot}1) = 1{\cdot}347,$$

and hence from (33)

$$\delta(9, 4{\cdot}743, 0{\cdot}1) = 4{\cdot}743 - (1{\cdot}347)(0{\cdot}6667) = 2{\cdot}723.$$

To obtain $\delta(9, 4{\cdot}743, 0{\cdot}9)$ we note by (36) that this is equivalent to *minus* $\delta(9, -4{\cdot}743, 0{\cdot}1)$. The corresponding λ is obtained by entering the same table as before, with the same value of y, but in the part of the table for which t_0 is negative. We obtain

$$\lambda(9, -4{\cdot}743, 0{\cdot}1) = 1{\cdot}197.$$

Hence $$\delta(9, -4{\cdot}743, 0{\cdot}1) = -4{\cdot}743 - (1{\cdot}197)(0{\cdot}6667) = -6{\cdot}538,$$

and therefore $\delta(9, 4{\cdot}743, 0{\cdot}9) = 6{\cdot}538$.

The upper and lower 10 % limits for U are therefore $6{\cdot}538/\sqrt{10}$ and $2{\cdot}723/\sqrt{10}$, i.e. 2·068 and 0·861. The corresponding limits for P are 0·019 and 0·195.

Example 3. Suppose it is required to estimate fiducial limits for the value L of x in the normal population which is such that the proportion of the population exceeding L is 10 %. Consider the case where the sample size is $n = 10$ and the limits required are the 5 % limits at each end. From (18), we require

$$\bar{x} + k_{0{\cdot}05}\, s \quad \text{and} \quad \bar{x} + k_{0{\cdot}95}\, s,$$

where $$k_\epsilon = \frac{t(9, \sqrt{10}\, U, \epsilon)}{\sqrt{10}}, \qquad \text{.....(40)}$$

and U is the normal deviate exceeded with probability 0·1; i.e. $U = 1{\cdot}2816$. We therefore require $t(9, 4{\cdot}053, 0{\cdot}05)$ and $t(9, 4{\cdot}053, 0{\cdot}95)$. In the first place these will be derived from Table IV by the inverse method described above.

* For a note on the accuracy of linear interpolation see the end of Section 9.

Following out the successive steps to derive $t(9, 4{\cdot}053, 0{\cdot}05)$ we obtain

$$\text{(i)}\quad t_1 = \frac{4{\cdot}053 + (1{\cdot}6449)(1 + 0{\cdot}9126 - 0{\cdot}1503)^{\frac{1}{2}}}{(1 - 0{\cdot}1503)} = 7{\cdot}340,$$

$$\text{(ii)}\quad y_1 = \left(1 + \frac{t_1^2}{2f}\right)^{-\frac{1}{2}} = 0{\cdot}5004; \quad \lambda_1 = \lambda(9,\ 7{\cdot}340,\ 0{\cdot}05) = 1{\cdot}6803,$$

$$\text{(iii)}\quad t_2 = \frac{4{\cdot}053 + (1{\cdot}6803)(1 + 0{\cdot}9126 - 0{\cdot}1569)^{\frac{1}{2}}}{(1 - 0{\cdot}1569)} = 7{\cdot}448,$$

$$\text{(iv)}\quad y_2 = \left(1 + \frac{t_2^2}{2f}\right)^{-\frac{1}{2}} = 0{\cdot}4950; \quad \lambda_2 = \lambda(9,\ 7{\cdot}448,\ 0{\cdot}05) = 1{\cdot}6800,$$
$$t_3 = 7{\cdot}447.$$

To obtain $t(9, 4{\cdot}053, 0{\cdot}95)$ we require first $t(9, -4{\cdot}053, 0{\cdot}05)$. The successive approximations to this quantity are

$$\text{(i)}\quad t_1 = \frac{-4{\cdot}053 + (1{\cdot}6449)(1 + 0{\cdot}9126 - 0{\cdot}1503)^{\frac{1}{2}}}{(1 - 0{\cdot}1503)} = -2{\cdot}200,$$

$$\text{(ii)}\quad y_1' = \frac{t_1}{\sqrt{(2f)}}\left(1 + \frac{t_1^2}{2f}\right)^{-\frac{1}{2}} = -0{\cdot}4602; \quad \lambda_1 = \lambda(9,\ -2{\cdot}200,\ 0{\cdot}05) = 1{\cdot}5933,$$

$$\text{(iii)}\quad t_2 = -2{\cdot}249,$$

$$\text{(iv)}\quad y_2' = \frac{t_2}{\sqrt{(2f)}}\left(1 + \frac{t_2^2}{2f}\right)^{-\frac{1}{2}} = -0{\cdot}4684; \quad \lambda_2 = \lambda(9,\ -2{\cdot}249,\ 0{\cdot}05) = 1{\cdot}5925,$$
$$t_3 = -2{\cdot}250;$$

$t(9, 4{\cdot}053, 0{\cdot}95)$ is equal to *minus* $t(9, -4{\cdot}053, 0{\cdot}05)$ and therefore equals $2{\cdot}250$.

Hence from (40)

$$k_{0{\cdot}05} = \frac{7{\cdot}447}{\sqrt{10}} = 2{\cdot}355; \quad k_{0{\cdot}95} = \frac{2{\cdot}250}{\sqrt{10}} = 0{\cdot}712.$$

The above calculations have been performed using Table IV and the inverse method described in the previous section. Actually this example could have been treated much more simply by Table V. We shall proceed to show this, but it must be remembered that Table V only covers cases where $\epsilon = 0{\cdot}05$ and $\epsilon = 0{\cdot}95$. For other probability levels Table IV will have to be used.

To obtain $t(9, 4{\cdot}053, 0{\cdot}05)$ and $t(9, -4{\cdot}053, 0{\cdot}05)$ Table V has to be entered with $\eta' = \dfrac{\delta}{\sqrt{(2f)}}\left(1 + \dfrac{\delta^2}{2f}\right)^{-\frac{1}{2}}$, where δ has the two values $\pm 4{\cdot}053$, i.e. has to be entered with $\eta' = \pm 0{\cdot}6908$. These give values of λ equal to $1{\cdot}6800$ and $1{\cdot}5925$ and are the same as the final λ's obtained in the iterations. We have then only to perform the calculations for t by substituting in (37).

9. Summary

An account of some of the applications of the non-central t-distribution* has been given. Tables of the distribution have been provided together with numerical examples of their use.

If t_0 and δ are connected by the equation $P(t > t_0 \mid \delta) = \epsilon$, then it was found that a good approximation to δ, given t_0, is

$$\delta = t_0 - K_\epsilon \left(1 + \frac{t_0^2}{2f}\right)^{\frac{1}{2}},$$

where K_ϵ is the normal deviate exceeded with probability ϵ. The equivalent approximation

$$t_0 = \frac{\delta + K_\epsilon \left(1 + \frac{\delta^2}{2f} - \frac{K_\epsilon^2}{2f}\right)^{\frac{1}{2}}}{\left(1 - \frac{K_\epsilon^2}{2f}\right)}$$

is good for calculating t_0 given δ. These approximations were seen to have advantages over other approximations also based only on the normal distribution.

Tables IV and V at the end of the paper make possible a more exact determination of δ given t_0 and t_0 given δ, respectively. In these tables a quantity λ_ϵ is given such that

$$\delta = t_0 - \lambda_\epsilon \left(1 + \frac{t_0^2}{2f}\right)^{\frac{1}{2}},$$

and so

$$t_0 = \frac{\delta + \lambda_\epsilon \left(1 + \frac{\delta^2}{2f} - \frac{\lambda_\epsilon^2}{2f}\right)^{\frac{1}{2}}}{\left(1 - \frac{\lambda_\epsilon^2}{2f}\right)}.$$

Note on the accuracy of the tables. In the greater part of the tables λ_ϵ is correct to as many figures as are shown. Occasionally, however, the values of λ_ϵ may be almost 2 units wrong in the last figure. It was nevertheless considered worth while to give all the figures shown.

Throughout Table IV linear interpolation will always give a result not more than a $\frac{1}{2}$ unit wrong in the second last figure. This is also true of Table V, except between $\eta' = -0{\cdot}8$ and $-1{\cdot}0$ for $f = 9$ and 16. Here linear interpolation with respect to η' may be 1 unit wrong in the second last figure.

* $t = (z+\delta)/\sqrt{w}$, where z is a unit normal deviate, w is distributed independently as χ^2/f, and f is the number of degrees of freedom of χ^2.

TABLE IV

Values of $\lambda(f, t_0, \epsilon)$

$\epsilon = 0{\cdot}005$

t_0		y'	y	$f=4$	5	6	7	8	9	16	36	144	∞
									$12/\sqrt{f}=4$	3	2	1	0
$-\infty$		$-1{\cdot}0$	0·0	2·62	2·62	2·63	2·63	2·63	2·626	2·622	2·613	2·598	2·576
			0·1	2·62	2·62	2·62	2·62	2·62	2·624	2·620	2·612	2·597	2·576
			0·2	2·61	2·62	2·62	2·62	2·62	2·619	2·616	2·608	2·595	2·576
			0·3	2·60	2·61	2·61	2·61	2·61	2·610	2·609	2·603	2·592	2·576
			0·4	2·59	2·59	2·60	2·60	2·60	2·599	2·599	2·596	2·588	2·576
			0·5	2·57	2·58	2·58	2·58	2·58	2·585	2·588	2·588	2·584	2·576
			0·6	2·55	2·56	2·56	2·57	2·57	2·569	2·575	2·578	2·579	2·576
			0·7	2·53	2·54	2·54	2·55	2·55	2·554	2·562	2·568	2·574	2·576
	Negative	$-0{\cdot}6$	0·8	2·52	2·52	2·53	2·53	2·54	2·541	2·552	2·561	2·569	2·576
		$-0{\cdot}6$	0·8	2·52	2·52	2·53	2·53	2·54	2·541	2·552	2·561	2·569	2·576
		$-0{\cdot}5$		2·51	2·52	2·53	2·53	2·53	2·537	2·548	2·558	2·568	2·576
		$-0{\cdot}4$		2·52	2·52	2·53	2·53	2·54	2·538	2·548	2·558	2·567	2·576
		$-0{\cdot}3$		2·53	2·53	2·54	2·54	2·54	2·544	2·552	2·560	2·568	2·576
		$-0{\cdot}2$		2·54	2·54	2·55	2·55	2·55	2·553	2·559	2·565	2·570	2·576
		$-0{\cdot}1$		2·56	2·56	2·56	2·56	2·56	2·564	2·567	2·570	2·573	2·576
0		0·0	1·0	2·58	2·58	2·58	2·58	2·58	2·576	2·576	2·576	2·576	2·576
		0·1		2·59	2·59	2·59	2·59	2·59	2·587	2·584	2·581	2·579	2·576
		0·2		2·60	2·60	2·60	2·60	2·60	2·596	2·591	2·586	2·581	2·576
		0·3		2·61	2·61	2·61	2·60	2·60	2·601	2·596	2·590	2·583	2·576
		0·4		2·61	2·61	2·61	2·60	2·60	2·602	2·597	2·591	2·584	2·576
		0·5		2·60	2·60	2·60	2·60	2·60	2·596	2·593	2·589	2·583	2·576
		0·6	0·8	2·57	2·57	2·58	2·58	2·58	2·582	2·584	2·583	2·581	2·576
	Positive	0·6	0·8	2·57	2·57	2·58	2·58	2·58	2·582	2·584	2·583	2·581	2·576
			0·7	2·52	2·53	2·54	2·54	2·55	2·552	2·564	2·571	2·575	2·576
			0·6	2·46	2·48	2·49	2·50	2·51	2·518	2·541	2·558	2·569	2·576
			0·5	2·39	2·42	2·45	2·46	2·47	2·484	2·517	2·544	2·563	2·576
			0·4	2·33	2·37	2·40	2·42	2·44	2·451	2·495	2·531	2·557	2·576
			0·3	2·27	2·33	2·36	2·39	2·41	2·422	2·477	2·520	2·552	2·576
			0·2	2·23	2·29	2·33	2·36	2·38	2·399	2·462	2·511	2·548	2·576
			0·1	2·20	2·27	2·31	2·34	2·37	2·385	2·453	2·506	2·546	2·576
∞		1·0	0·0	2·19	2·26	2·30	2·34	2·36	2·380	2·450	2·504	2·545	2·576

$$y = \left(1 + \frac{t_0^2}{2f}\right)^{-\frac{1}{2}};$$

$$y' = \frac{t_0}{\sqrt{(2f)}}\left(1 + \frac{t_0^2}{2f}\right)^{-\frac{1}{2}}.$$

For note on accuracy of table see end of Section 9.

$$\delta(f, t_0, \epsilon) = t_0 - \lambda(f, t_0, \epsilon)\left(1 + \frac{t_0^2}{2f}\right)^{\frac{1}{2}}; \qquad t(f, \delta, \epsilon) = \frac{\delta + \lambda\left(1 + \frac{\delta^2}{2f} - \frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1 - \frac{\lambda^2}{2f}\right)}.$$

TABLE IV (*continued*)

Values of $\lambda(f, t_0, \epsilon)$

$\epsilon = 0{\cdot}01$

t_0	y'	y	$f=4$	5	6	7	8	9	16	36	144	∞
								$12/\sqrt{f}=4$	3	2	1	0
$-\infty$	$-1{\cdot}0$	0·0	2·32	2·33	2·33	2·34	2·34	2·340	2·343	2·342	2·337	2·326
		0·1	2·32	2·33	2·33	2·34	2·34	2·339	2·342	2·342	2·337	2·326
		0·2	2·32	2·32	2·33	2·33	2·33	2·335	2·339	2·339	2·335	2·326
		0·3	2·31	2·32	2·32	2·32	2·33	2·328	2·334	2·335	2·333	2·326
		0·4	2·30	2·31	2·31	2·32	2·32	2·320	2·327	2·330	2·330	2·326
		0·5	2·29	2·30	2·30	2·30	2·31	2·310	2·318	2·324	2·327	2·326
		0·6	2·27	2·28	2·29	2·29	2·30	2·299	2·310	2·318	2·323	2·326
		0·7	2·26	2·27	2·28	2·28	2·29	2·289	2·301	2·311	2·320	2·326
Negative	$-0{\cdot}6$	0·8	2·25	2·26	2·27	2·28	2·28	2·282	2·295	2·307	2·317	2·326
	$-0{\cdot}6$	0·8	2·25	2·26	2·27	2·28	2·28	2·282	2·295	2·307	2·317	2·326
	$-0{\cdot}5$		2·26	2·26	2·27	2·28	2·28	2·282	2·294	2·306	2·317	2·326
	$-0{\cdot}4$		2·26	2·27	2·28	2·28	2·28	2·286	2·297	2·307	2·317	2·326
	$-0{\cdot}3$		2·28	2·28	2·29	2·29	2·29	2·293	2·302	2·310	2·319	2·326
	$-0{\cdot}2$		2·29	2·30	2·30	2·30	2·30	2·303	2·309	2·315	2·321	2·326
	$-0{\cdot}1$		2·31	2·31	2·31	2·31	2·31	2·315	2·318	2·320	2·323	2·326
0	0·0	1·0	2·33	2·33	2·33	2·33	2·33	2·326	2·326	2·326	2·326	2·326
	0·1		2·34	2·34	2·34	2·34	2·34	2·337	2·335	2·332	2·329	2·326
	0·2		2·35	2·35	2·35	2·35	2·35	2·347	2·342	2·337	2·332	2·326
	0·3		2·36	2·36	2·36	2·36	2·35	2·353	2·347	2·341	2·334	2·326
	0·4		2·36	2·36	2·36	2·36	2·36	2·356	2·350	2·343	2·335	2·326
	0·5		2·36	2·36	2·36	2·36	2·35	2·353	2·349	2·343	2·335	2·326
	0·6	0·8	2·34	2·34	2·34	2·34	2·34	2·344	2·343	2·340	2·334	2·326
Positive	0·6	0·8	2·34	2·34	2·34	2·34	2·34	2·344	2·343	2·340	2·334	2·326
		0·7	2·30	2·31	2·32	2·32	2·32	2·324	2·329	2·332	2·330	2·326
		0·6	2·26	2·27	2·28	2·29	2·30	2·299	2·313	2·322	2·326	2·326
		0·5	2·21	2·23	2·25	2·26	2·27	2·274	2·296	2·312	2·322	2·326
		0·4	2·16	2·19	2·21	2·23	2·24	2·250	2·280	2·302	2·317	2·326
		0·3	2·12	2·16	2·19	2·20	2·22	2·229	2·267	2·294	2·314	2·326
		0·2	2·09	2·13	2·16	2·19	2·20	2·213	2·256	2·288	2·311	2·326
		0·1	2·07	2·12	2·15	2·17	2·19	2·203	2·250	2·285	2·309	2·326
∞	1·0	0·0	2·06	2·11	2·15	2·17	2·19	2·199	2·247	2·283	2·309	2·326

$$y = \left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}};$$

$$y' = \frac{t_0}{\sqrt{(2f)}}\left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}}.$$

For note on accuracy of table see end of Section 9.

$$\delta(f, t_0, \epsilon) = t_0 - \lambda(f, t_0, \epsilon)\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{2}}; \qquad t(f, \delta, \epsilon) = \frac{\delta + \lambda\left(1+\frac{\delta^2}{2f}-\frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1-\frac{\lambda^2}{2f}\right)}.$$

TABLE IV (*continued*)

Values of $\lambda(f, t_0, \epsilon)$

TABLE IV (*continued*)

Values of $\lambda(f, t_0, \epsilon)$

$\epsilon = 0{\cdot}025$

t_0	y'	y	$f=4$	5	6	7	8	9	16	36	144	∞
								$12/\sqrt{f}=4$	3	2	1	0
$-\infty$	$-1{\cdot}0$	0·0	1·892	1·904	1·912	1·917	1·922	1·925	1·939	1·950	1·956	1·960
		0·1	1·891	1·903	1·911	1·917	1·921	1·925	1·938	1·949	1·956	1·960
		0·2	1·889	1·900	1·908	1·914	1·919	1·923	1·936	1·947	1·955	1·960
		0·3	1·885	1·897	1·905	1·911	1·916	1·919	1·934	1·945	1·954	1·960
		0·4	1·880	1·892	1·900	1·907	1·912	1·915	1·930	1·943	1·953	1·960
		0·5	1·875	1·887	1·896	1·902	1·907	1·911	1·926	1·940	1·951	1·960
		0·6	1·871	1·883	1·891	1·898	1·903	1·907	1·923	1·937	1·949	1·960
		0·7	1·869	1·880	1·889	1·895	1·900	1·904	1·920	1·935	1·948	1·960
Negative	$-0{\cdot}6$	0·8	1·872	1·883	1·890	1·896	1·901	1·905	1·920	1·934	1·948	1·960
	$-0{\cdot}6$	0·8	1·872	1·883	1·890	1·896	1·901	1·905	1·920	1·934	1·948	1·960
	$-0{\cdot}5$		1·881	1·890	1·896	1·902	1·906	1·909	1·923	1·936	1·948	1·960
	$-0{\cdot}4$		1·893	1·900	1·906	1·910	1·913	1·916	1·928	1·939	1·950	1·960
	$-0{\cdot}3$		1·908	1·913	1·918	1·921	1·924	1·926	1·935	1·943	1·952	1·960
	$-0{\cdot}2$		1·925	1·928	1·931	1·933	1·935	1·937	1·943	1·948	1·954	1·960
	$-0{\cdot}1$		1·942	1·944	1·946	1·947	1·948	1·948	1·951	1·954	1·957	1·960
0	0·0	1·0	1·960	1·960	1·960	1·960	1·960	1·960	1·960	1·960	1·960	1·960
	0·1		1·976	1·975	1·974	1·973	1·972	1·971	1·968	1·966	1·963	1·960
	0·2		1·990	1·987	1·985	1·984	1·982	1·981	1·976	1·971	1·966	1·960
	0·3		2·000	1·997	1·994	1·992	1·991	1·989	1·982	1·975	1·968	1·960
	0·4		2·006	2·003	2·000	1·998	1·996	1·994	1·987	1·979	1·970	1·960
	0·5		2·006	2·004	2·001	2·000	1·998	1·996	1·989	1·981	1·971	1·960
	0·6	0·8	1·998	1·998	1·997	1·996	1·995	1·994	1·988	1·980	1·971	1·960
Positive	0·6	0·8	1·998	1·998	1·997	1·996	1·995	1·994	1·988	1·980	1·971	1·960
		0·7	1·978	1·982	1·983	1·984	1·984	1·985	1·983	1·978	1·970	1·960
		0·6	1·953	1·961	1·966	1·969	1·971	1·972	1·975	1·973	1·968	1·960
		0·5	1·926	1·939	1·947	1·952	1·956	1·959	1·966	1·968	1·966	1·960
		0·4	1·900	1·918	1·929	1·936	1·942	1·946	1·957	1·963	1·964	1·960
		0·3	1·878	1·900	1·913	1·923	1·930	1·935	1·950	1·959	1·962	1·960
		0·2	1·860	1·885	1·901	1·912	1·920	1·926	1·945	1·956	1·960	1·960
		0·1	1·849	1·876	1·893	1·906	1·914	1·921	1·941	1·954	1·959	1·960
∞	1·0	0·0	1·844	1·872	1·890	1·903	1·912	1·919	1·940	1·953	1·959	1·960

$$y = \left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}};$$

$$y' = \frac{t_0}{\sqrt{(2f)}}\left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}}.$$

For note on accuracy of table see end of Section 9.

$$\delta(f, t_0, \epsilon) = t_0 - \lambda(f, t_0, \epsilon)\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{2}}; \qquad t(f, \delta, \epsilon) = \frac{\delta + \lambda\left(1+\frac{\delta^2}{2f}-\frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1-\frac{\lambda^2}{2f}\right)}.$$

TABLE IV (*continued*)

Values of $\lambda(f, t_0, \epsilon)$

$\epsilon = 0{\cdot}05$

t_0	y'	y	$f=4$	5	6	7	8	9	16	36	144	∞
								$12/\sqrt{f}=4$	3	2	1	0
$-\infty$	$-1{\cdot}0$	0·0	1·528	1·543	1·554	1·563	1·569	1·5744	1·5952	1·6141	1·6307	1·6449
		0·1	1·527	1·543	1·554	1·562	1·569	1·5742	1·5950	1·6139	1·6306	1·6449
		0·2	1·527	1·542	1·553	1·562	1·568	1·5736	1·5945	1·6135	1·6303	1·6449
		0·3	1·526	1·542	1·553	1·561	1·567	1·5728	1·5937	1·6129	1·6300	1·6449
		0·4	1·526	1·541	1·552	1·560	1·567	1·5720	1·5930	1·6122	1·6295	1·6449
		0·5	1·526	1·541	1·552	1·560	1·566	1·5717	1·5924	1·6116	1·6291	1·6449
		0·6	1·529	1·543	1·553	1·561	1·567	1·5724	1·5926	1·6115	1·6289	1·6449
		0·7	1·534	1·548	1·557	1·565	1·570	1·5751	1·5942	1·6123	1·6292	1·6449
Negative	$-0{\cdot}6$	0·8	1·546	1·557	1·566	1·572	1·577	1·5816	1·5986	1·6149	1·6303	1·6449
	$-0{\cdot}6$	0·8	1·546	1·557	1·566	1·572	1·577	1·5816	1·5986	1·6149	1·6303	1·6449
	$-0{\cdot}5$		1·559	1·569	1·576	1·581	1·586	1·5895	1·6041	1·6183	1·6319	1·6449
	$-0{\cdot}4$		1·575	1·582	1·588	1·593	1·596	1·5990	1·6110	1·6226	1·6339	1·6449
	$-0{\cdot}3$		1·592	1·597	1·602	1·605	1·608	1·6097	1·6187	1·6276	1·6363	1·6449
	$-0{\cdot}2$		1·609	1·613	1·616	1·618	1·620	1·6212	1·6272	1·6331	1·6390	1·6449
	$-0{\cdot}1$		1·627	1·629	1·630	1·632	1·632	1·6332	1·6360	1·6390	1·6419	1·6449
0	0·0	1·0	1·645	1·645	1·645	1·645	1·645	1·6449	1·6449	1·6449	1·6449	1·6449
	0·1		1·661	1·660	1·658	1·658	1·657	1·6561	1·6534	1·6506	1·6478	1·6449
	0·2		1·676	1·673	1·671	1·669	1·668	1·6665	1·6614	1·6561	1·6506	1·6449
	0·3		1·688	1·684	1·681	1·679	1·677	1·6756	1·6686	1·6610	1·6531	1·6449
	0·4		1·697	1·693	1·690	1·687	1·685	1·6830	1·6745	1·6652	1·6553	1·6449
	0·5		1·702	1·698	1·695	1·692	1·690	1·6881	1·6789	1·6685	1·6571	1·6449
	0·6	0·8	1·703	1·700	1·697	1·695	1·693	1·6906	1·6815	1·6707	1·6584	1·6449
Positive	0·6	0·8	1·703	1·700	1·697	1·695	1·693	1·6906	1·6815	1·6707	1·6584	1·6449
		0·7	1·696	1·695	1·694	1·692	1·691	1·6895	1·6817	1·6714	1·6590	1·6449
		0·6	1·685	1·687	1·687	1·687	1·686	1·6855	1·6797	1·6706	1·6588	1·6449
		0·5	1·673	1·677	1·680	1·680	1·680	1·6803	1·6767	1·6691	1·6583	1·6449
		0·4	1·661	1·668	1·672	1·674	1·675	1·6750	1·6735	1·6674	1·6576	1·6449
		0·3	1·651	1·660	1·665	1·668	1·670	1·6704	1·6706	1·6658	1·6569	1·6449
		0·2	1·643	1·654	1·660	1·663	1·666	1·6669	1·6684	1·6645	1·6564	1·6449
		0·1	1·639	1·650	1·657	1·661	1·663	1·6646	1·6670	1·6637	1·6560	1·6449
∞	1·0	0·0	1·636	1·648	1·655	1·660	1·662	1·6638	1·6665	1·6634	1·6559	1·6449

$$y = \left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}};$$

$$y' = \frac{t_0}{\sqrt{(2f)}}\left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}}.$$

For note on accuracy of table see end of Section 9.

$$\delta(f, t_0, \epsilon) = t_0 - \lambda(f, t_0, \epsilon)\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{2}}; \qquad t(f, \delta, \epsilon) = \frac{\delta + \lambda\left(1+\frac{\delta^2}{2f}-\frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1-\frac{\lambda^2}{2f}\right)}.$$

TABLE IV (*continued*)

$$\delta(f, t_0, \epsilon) = t_0 - \lambda(f, t_0, \epsilon)\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{2}}; \qquad t(f, \delta, \epsilon) = \frac{\delta + \lambda\left(1+\frac{\delta^2}{2f}-\frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1-\frac{\lambda^2}{2f}\right)}.$$

TABLE IV (*continued*)

Values of $\lambda(f, t_0, \epsilon)$

$\epsilon = 0{\cdot}10$

t_0		y'	y	$f=4$	5	6	7	8	9	16	36	144	∞
									$12/\sqrt{f}=4$	3	2	1	0
$-\infty$		$-1{\cdot}0$	0·0	1·116	1·136	1·150	1·161	1·169	1·1765	1·2049	1·2319	1·2575	1·2816
			0·1	1·116	1·136	1·150	1·161	1·170	1·1768	1·2051	1·2321	1·2576	1·2816
			0·2	1·118	1·137	1·151	1·162	1·171	1·1777	1·2057	1·2325	1·2578	1·2816
			0·3	1·121	1·140	1·154	1·164	1·172	1·1793	1·2069	1·2332	1·2581	1·2816
			0·4	1·125	1·143	1·157	1·167	1·175	1·1819	1·2087	1·2343	1·2586	1·2816
			0·5	1·131	1·149	1·162	1·171	1·179	1·1856	1·2114	1·2360	1·2594	1·2816
			0·6	1·140	1·157	1·169	1·178	1·185	1·1912	1·2153	1·2385	1·2606	1·2816
			0·7	1·153	1·168	1·179	1·187	1·194	1·1992	1·2210	1·2421	1·2623	1·2816
	Negative	$-0{\cdot}6$	0·8	1·173	1·185	1·194	1·201	1·206	1·2110	1·2295	1·2475	1·2649	1·2816
		$-0{\cdot}6$	0·8	1·173	1·185	1·194	1·201	1·206	1·2110	1·2295	1·2475	1·2649	1·2816
		$-0{\cdot}5$		1·191	1·201	1·208	1·214	1·218	1·2222	1·2376	1·2527	1·2673	1·2816
		$-0{\cdot}4$		1·209	1·217	1·223	1·227	1·231	1·2338	1·2461	1·2582	1·2700	1·2816
		$-0{\cdot}3$		1·228	1·233	1·238	1·241	1·244	1·2458	1·2548	1·2639	1·2728	1·2816
		$-0{\cdot}2$		1·246	1·250	1·253	1·255	1·256	1·2578	1·2638	1·2697	1·2757	1·2816
		$-0{\cdot}1$		1·264	1·266	1·267	1·268	1·269	1·2698	1·2727	1·2756	1·2786	1·2816
0		0·0	1·0	1·282	1·282	1·282	1·282	1·282	1·2816	1·2816	1·2816	1·2816	1·2816
		0·1		1·298	1·297	1·295	1·294	1·294	1·2929	1·2902	1·2874	1·2845	1·2816
		0·2		1·313	1·310	1·308	1·306	1·305	1·3038	1·2985	1·2930	1·2873	1·2816
		0·3		1·327	1·323	1·320	1·318	1·316	1·3140	1·3064	1·2984	1·2901	1·2816
		0·4		1·340	1·335	1·331	1·328	1·326	1·3233	1·3137	1·3035	1·2927	1·2816
		0·5		1·350	1·345	1·341	1·337	1·334	1·3316	1·3204	1·3082	1·2952	1·2816
		0·6	0·8	1·358	1·353	1·349	1·345	1·342	1·3387	1·3263	1·3124	1·2974	1·2816
		0·6	0·8	1·358	1·353	1·349	1·345	1·342	1·3387	1·3263	1·3124	1·2974	1·2816
			0·7	1·365	1·360	1·355	1·352	1·348	1·3453	1·3320	1·3166	1·2997	1·2816
			0·6	1·367	1·363	1·359	1·355	1·352	1·3492	1·3354	1·3193	1·3012	1·2816
			0·5	1·368	1·365	1·361	1·358	1·354	1·3516	1·3377	1·3210	1·3021	1·2816
			0·4	1·369	1·366	1·362	1·359	1·356	1·3531	1·3392	1·3222	1·3028	1·2816
			0·3	1·369	1·367	1·363	1·360	1·357	1·3541	1·3401	1·3229	1·3032	1·2816
			0·2	1·370	1·367	1·364	1·361	1·358	1·3548	1·3408	1·3234	1·3035	1·2816
			0·1	1·370	1·367	1·364	1·361	1·358	1·3552	1·3411	1·3237	1·3037	1·2816
∞		1·0	0·0	1·370	1·368	1·364	1·361	1·358	1·3554	1·3413	1·3238	1·3038	1·2816

$$y = \left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}};$$

$$y' = \frac{t_0}{\sqrt{(2f)}}\left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}}.$$

For note on accuracy of table see end of Section 9.

$$\delta(f, t_0, \epsilon) = t_0 - \lambda(f, t_0, \epsilon)\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{2}}; \qquad t(f, \delta, \epsilon) = \frac{\delta + \lambda\left(1+\frac{\delta^2}{2f}-\frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1-\frac{\lambda^2}{2f}\right)}.$$

TABLE IV (*continued*)

Values of $\lambda(f, t_0, \epsilon)$

$\epsilon = 0{\cdot}20$

t_0	y'	y	$f=4$	5	6	7	8	9	16	36	144	∞
								$12/\sqrt{f}=4$	3	2	1	0
$-\infty$	$-1{\cdot}0$	0·0	·632	·656	·673	·686	·697	·7055	·7408	·7753	·8089	·8416
		0·1	·633	·657	·674	·687	·698	·7063	·7414	·7756	·8091	·8416
		0·2	·637	·660	·677	·690	·700	·7086	·7431	·7768	·8096	·8416
		0·3	·643	·665	·682	·694	·704	·7125	·7460	·7787	·8106	·8416
		0·4	·652	·673	·689	·701	·710	·7182	·7503	·7815	·8120	·8416
		0·5	·664	·684	·698	·710	·718	·7259	·7559	·7853	·8138	·8416
		0·6	·679	·697	·711	·721	·729	·7359	·7633	·7901	·8162	·8416
		0·7	·699	·715	·727	·736	·743	·7487	·7727	·7962	·8192	·8416
Negative	$-0{\cdot}6$	0·8	·725	·738	·747	·755	·760	·7652	·7848	·8041	·8231	·8416
	$-0{\cdot}6$	0·8	·725	·738	·747	·755	·760	·7652	·7848	·8041	·8231	·8416
	$-0{\cdot}5$		·747	·757	·765	·771	·775	·7791	·7950	·8108	·8264	·8416
	$-0{\cdot}4$		·768	·776	·781	·786	·790	·7925	·8049	·8173	·8295	·8416
	$-0{\cdot}3$		·788	·793	·797	·801	·803	·8054	·8145	·8236	·8326	·8416
	$-0{\cdot}2$		·806	·810	·813	·815	·816	·8178	·8238	·8297	·8357	·8416
	$-0{\cdot}1$		·824	·826	·827	·828	·829	·8299	·8328	·8357	·8387	·8416
0	0·0	1·0	·842	·842	·842	·842	·842	·8416	·8416	·8416	·8416	·8416
	0·1		·858	·857	·856	·855	·854	·8531	·8503	·8475	·8446	·8416
	0·2		·875	·872	·869	·867	·866	·8644	·8589	·8533	·8475	·8416
	0·3		·890	·886	·883	·880	·878	·8756	·8675	·8591	·8505	·8416
	0·4		·906	·900	·896	·892	·889	·8868	·8762	·8650	·8535	·8416
	0·5		·921	·914	·909	·905	·901	·8980	·8849	·8710	·8565	·8416
	0·6	0·8	·937	·929	·922	·917	·913	·9094	·8937	·8771	·8596	·8416
Positive	0·6	0·8	·937	·929	·922	·917	·913	·9094	·8937	·8771	·8596	·8416
		0·7	·955	·946	·938	·932	·927	·9229	·9042	·8843	·8633	·8416
		0·6	·970	·959	·951	·944	·939	·9337	·9125	·8899	·8662	·8416
		0·5	·982	·971	·961	·954	·948	·9424	·9192	·8945	·8685	·8416
		0·4	·992	·980	·970	·962	·955	·9495	·9246	·8981	·8703	·8416
		0·3	1·001	·987	·977	·968	·961	·9550	·9288	·9008	·8717	·8416
		0·2	1·007	·993	·982	·973	·965	·9591	·9317	·9028	·8726	·8416
		0·1	1·011	·996	·985	·976	·968	·9615	·9335	·9040	·8732	·8416
∞	1·0	0·0	1·012	·997	·986	·977	·969	·9623	·9341	·9044	·8734	·8416

$$y=\left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}};$$

$$y'=\frac{t_0}{\sqrt{(2f)}}\left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}}.$$

For note on accuracy of table see end of Section 9.

$$\delta(f, t_0, \epsilon)=t_0-\lambda(f, t_0, \epsilon)\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{2}}; \qquad t(f, \delta, \epsilon)=\frac{\delta+\lambda\left(1+\frac{\delta^2}{2f}-\frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1-\frac{\lambda^2}{2f}\right)}.$$

TABLE IV (*continued*)

TABLE IV (*continued*)

Values of $\lambda(f, t_0, \epsilon)$

$\epsilon = 0{\cdot}30$

t_0		y'	y	$f=4$	5	6	7	8	9	16	36	144	∞
									$12/\sqrt{f}=4$	3	2	1	0
$-\infty$		$-1{\cdot}0$	0·0	·295	·320	·339	·353	·365	·3739	·4124	·4503	·4876	·5244
			0·1	·297	·322	·340	·354	·366	·3749	·4132	·4508	·4878	·5244
			0·2	·301	·326	·344	·358	·369	·3780	·4154	·4523	·4886	·5244
			0·3	·309	·333	·350	·364	·374	·3831	·4193	·4548	·4899	·5244
			0·4	·320	·343	·359	·372	·382	·3904	·4247	·4584	·4917	·5244
			0·5	·335	·356	·371	·383	·392	·4000	·4318	·4632	·4940	·5244
			0·6	·354	·373	·386	·397	·405	·4122	·4408	·4691	·4970	·5244
			0·7	·377	·393	·405	·414	·421	·4272	·4519	·4764	·5006	·5244
	Negative	$-0{\cdot}6$	0·8	·406	·419	·428	·435	·441	·4458	·4656	·4854	·5050	·5244
		$-0{\cdot}6$	0·8	·406	·419	·428	·435	·441	·4458	·4656	·4854	·5050	·5244
		$-0{\cdot}5$		·429	·439	·446	·452	·457	·4608	·4768	·4928	·5087	·5244
		$-0{\cdot}4$		·451	·458	·464	·468	·472	·4749	·4873	·4997	·5121	·5244
		$-0{\cdot}3$		·471	·476	·480	·483	·486	·4881	·4971	·5062	·5153	·5244
		$-0{\cdot}2$		·489	·493	·496	·498	·499	·5007	·5065	·5125	·5184	·5244
		$-0{\cdot}1$		·507	·509	·510	·511	·512	·5127	·5156	·5185	·5214	·5244
0		0·0	1·0	·524	·524	·524	·524	·524	·5244	·5244	·5244	·5244	·5244
		0·1		·541	·540	·538	·537	·537	·5360	·5331	·5303	·5273	·5244
		0·2		·558	·555	·552	·550	·549	·5475	·5419	·5362	·5303	·5244
		0·3		·575	·570	·567	·564	·561	·5593	·5509	·5422	·5334	·5244
		0·4		·592	·586	·581	·577	·574	·5715	·5602	·5485	·5366	·5244
		0·5		·610	·602	·596	·591	·587	·5842	·5699	·5551	·5399	·5244
		0·6	0·8	·630	·620	·613	·607	·602	·5977	·5803	·5621	·5434	·5244
	Positive	0·6	0·8	·630	·620	·613	·607	·602	·5977	·5803	·5621	·5434	·5244
		0·7		·654	·642	·633	·626	·620	·6146	·5931	·5707	·5478	·5244
		0·6		·674	·660	·650	·641	·634	·6285	·6036	·5778	·5513	·5244
		0·5		·691	·676	·664	·654	·647	·6401	·6123	·5836	·5542	·5244
		0·4		·706	·689	·676	·665	·657	·6496	·6194	·5883	·5566	·5244
		0·3		·717	·699	·685	·674	·665	·6570	·6249	·5920	·5584	·5244
		0·2		·726	·706	·692	·680	·670	·6624	·6289	·5946	·5597	·5244
		0·1		·731	·711	·696	·684	·674	·6657	·6312	·5961	·5604	·5244
∞		1·0	0·0	·733	·712	·697	·685	·675	·6668	·6320	·5966	·5607	·5244

$$y = \left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}};$$

$$y' = \frac{t_0}{\sqrt{(2f)}}\left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}}.$$

For note on accuracy of table see end of Section 9.

$$\delta(f, t_0, \epsilon) = t_0 - \lambda(f, t_0, \epsilon)\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{2}}; \qquad t(f, \delta, \epsilon) = \frac{\delta + \lambda\left(1+\frac{\delta^2}{2f}-\frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1-\frac{\lambda^2}{2f}\right)}.$$

TABLE IV (*continued*)

Values of $\lambda(f, t_0, \epsilon)$

$\epsilon = 0{\cdot}40$

t_0	y'	y	$f=4$	5	6	7	8	9	16	36	144	∞
								$12/\sqrt{f}=4$	3	2	1	0
$-\infty$	$-1{\cdot}0$	0·0	·016	·041	·060	·075	·087	·0964	·1362	·1755	·2146	·2533
		0·1	·018	·043	·062	·076	·088	·0975	·1370	·1761	·2149	·2533
		0·2	·023	·048	·066	·080	·092	·1010	·1396	·1778	·2157	·2533
		0·3	·032	·056	·073	·087	·098	·1068	·1439	·1807	·2172	·2533
		0·4	·044	·067	·083	·096	·106	·1150	·1500	·1847	·2192	·2533
		0·5	·061	·081	·096	·108	·118	·1256	·1578	·1899	·2218	·2533
		0·6	·081	·099	·113	·123	·132	·1388	·1676	·1964	·2250	·2533
		0·7	·106	·121	·133	·142	·149	·1548	·1795	·2042	·2289	·2533
Negative	$-0{\cdot}6$	0·8	·136	·148	·157	·164	·169	·1742	·1940	·2138	·2336	·2533
	$-0{\cdot}6$	0·8	·136	·148	·157	·164	·169	·1742	·1940	·2138	·2336	·2533
	$-0{\cdot}5$		·159	·169	·176	·181	·186	·1897	·2055	·2215	·2374	·2533
	$-0{\cdot}4$		·180	·188	·193	·197	·201	·2039	·2162	·2285	·2409	·2533
	$-0{\cdot}3$		·200	·205	·209	·212	·215	·2172	·2261	·2352	·2442	·2533
	$-0{\cdot}2$		·219	·222	·225	·227	·228	·2297	·2355	·2414	·2474	·2533
	$-0{\cdot}1$		·236	·238	·239	·240	·241	·2417	·2445	·2474	·2504	·2533
0	0·0	1·0	·253	·253	·253	·253	·253	·2533	·2533	·2533	·2533	·2533
	0·1		·270	·269	·267	·266	·266	·2649	·2621	·2592	·2563	·2533
	0·2		·287	·284	·282	·280	·278	·2767	·2710	·2652	·2593	·2533
	0·3		·305	·300	·296	·293	·291	·2888	·2802	·2714	·2624	·2533
	0·4		·323	·317	·312	·308	·304	·3016	·2899	·2779	·2657	·2533
	0·5		·343	·335	·328	·323	·319	·3152	·3002	·2848	·2692	·2533
	0·6	0·8	·365	·354	·346	·340	·334	·3299	·3113	·2923	·2729	·2533
Positive	0·6	0·8	·365	·354	·346	·340	·334	·3299	·3113	·2923	·2729	·2533
		0·7	·392	·379	·369	·361	·354	·3485	·3254	·3017	·2776	·2533
		0·6	·416	·400	·388	·378	·371	·3641	·3370	·3094	·2815	·2533
		0·5	·435	·417	·404	·393	·384	·3771	·3467	·3158	·2847	·2533
		0·4	·452	·432	·417	·405	·396	·3877	·3546	·3211	·2873	·2533
		0·3	·465	·443	·427	·415	·404	·3960	·3607	·3251	·2893	·2533
		0·2	·474	·452	·435	·422	·411	·4019	·3651	·3280	·2907	·2533
		0·1	·480	·457	·439	·426	·415	·4052	·3677	·3297	·2916	·2533
∞	1·0	0·0	·482	·458	·441	·427	·416	·4067	·3686	·3303	·2919	·2533

$$y=\left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}};$$

$$y'=\frac{t_0}{\sqrt{(2f)}}\left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}}$$

For note on accuracy of table see end of Section 9.

$$\delta(f, t_0, \epsilon)=t_0-\lambda(f, t_0, \epsilon)\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{2}}; \qquad t(f, \delta, \epsilon)=\frac{\delta+\lambda\left(1+\frac{\delta^2}{2f}-\frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1-\frac{\lambda^2}{2f}\right)}.$$

TABLE IV (*continued*)

TABLE IV (*continued*)

Values of $\lambda(f, t_0, \epsilon)$

$\epsilon = 0{\cdot}50$

t_0	y'	y	$f=4$	5	6	7	8	9	16	36	144	∞
								$12/\sqrt{f}=4$	3	2	1	0
$-\infty$	$-1{\cdot}0$	0·0	−·237	−·212	−·194	−·179	−·167	−·1578	−·1182	−·0787	−·0393	·0000
		0·1	−·235	−·210	−·192	−·178	−·166	−·1566	−·1173	−·0781	−·0390	·0000
		0·2	−·230	−·205	−·188	−·174	−·162	−·1530	−·1146	−·0763	−·0381	·0000
		0·3	−·220	−·197	−·180	−·167	−·156	−·1469	−·1101	−·0733	−·0366	·0000
		0·4	−·207	−·185	−·169	−·157	−·147	−·1384	−·1038	−·0692	−·0345	·0000
		0·5	−·190	−·171	−·156	−·144	−·135	−·1275	−·0957	−·0638	−·0319	·0000
		0·6	−·169	−·152	−·139	−·129	−·121	−·1140	−·0857	−·0572	−·0286	·0000
		0·7	−·145	−·130	−·119	−·111	−·104	−·0979	−·0736	−·0492	−·0246	·0000
Negative	$-0{\cdot}6$	0·8	−·116	−·104	−·095	−·089	−·083	−·0784	−·0591	−·0395	−·0198	·0000
	$-0{\cdot}6$	0·8	−·116	−·104	−·095	−·089	−·083	−·0784	−·0591	−·0395	−·0198	·0000
	$-0{\cdot}5$		−·093	−·084	−·077	−·071	−·067	−·0631	−·0476	−·0318	−·0159	·0000
	$-0{\cdot}4$		−·072	−·065	−·060	−·055	−·052	−·0490	−·0370	−·0248	−·0124	·0000
	$-0{\cdot}3$		−·053	−·047	−·044	−·041	−·038	−·0359	−·0271	−·0181	−·0091	·0000
	$-0{\cdot}2$		−·034	−·031	−·029	−·027	−·025	−·0235	−·0178	−·0119	−·0060	·0000
	$-0{\cdot}1$		−·017	−·015	−·014	−·013	−·012	−·0116	−·0088	−·0059	−·0030	·0000
0	0·0	1·0	·000	·000	·000	·000	·000	·0000	·0000	·0000	·0000	·0000
	0·1		·017	·015	·014	·013	·012	·0116	·0088	·0059	·0030	·0000
	0·2		·034	·031	·029	·027	·025	·0235	·0178	·0119	·0060	·0000
	0·3		·053	·047	·044	·041	·038	·0359	·0271	·0181	·0091	·0000
	0·4		·072	·065	·060	·055	·052	·0490	·0370	·0248	·0124	·0000
	0·5		·093	·084	·077	·071	·067	·0631	·0476	·0318	·0159	·0000
	0·6	0·8	·116	·104	·095	·089	·083	·0784	·0591	·0395	·0198	·0000
Positive	0·6	0·8	·116	·104	·095	·089	·083	·0784	·0591	·0395	·0198	·0000
		0·7	·145	·130	·119	·111	·104	·0979	·0736	·0492	·0246	·0000
		0·6	·169	·152	·139	·129	·121	·1140	·0857	·0572	·0286	·0000
		0·5	·190	·171	·156	·144	·135	·1275	·0957	·0638	·0319	·0000
		0·4	·207	·185	·169	·157	·147	·1384	·1038	·0692	·0345	·0000
		0·3	·220	·197	·180	·167	·156	·1469	·1101	·0733	·0366	·0000
		0·2	·230	·205	·188	·174	·162	·1530	·1146	·0763	·0381	·0000
		0·1	·235	·210	·192	·178	·166	·1566	·1173	·0781	·0390	·0000
∞	1·0	0·0	·237	·212	·194	·179	·167	·1578	·1182	·0787	·0393	·0000

$$y = \left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}};$$

$$y' = \frac{t_0}{\sqrt{(2f)}}\left(1+\frac{t_0^2}{2f}\right)^{-\frac{1}{2}}.$$

For note on accuracy of table see end of Section 9.

$$\delta(f, t_0, \epsilon) = t_0 - \lambda(f, t_0, \epsilon)\left(1+\frac{t_0^2}{2f}\right)^{\frac{1}{2}}; \qquad \mathit{ll}(f, \delta, \epsilon) = \frac{\delta + \lambda\left(1+\frac{\delta^2}{2f}-\frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1-\frac{\lambda^2}{2f}\right)}.$$

TABLE V

Values of λ as a function of δ

$\epsilon = 0{\cdot}05$

η' \ f	4	5	6	7	8	9	16	36	144	∞
1·0	1·636	1·648	1·655	1·660	1·662	1·6638	1·6665	1·6634	1·6559	1·6449
0·9	1·643	1·655	1·662	1·666	1·668	1·6695	1·6711	1·6667	1·6576	1·6449
0·8	1·650	1·662	1·668	1·672	1·674	1·6747	1·6751	1·6691	1·6586	1·6449
0·7	1·657	1·668	1·674	1·677	1·679	1·6796	1·6782	1·6707	1·6589	1·6449
0·6	1·664	1·675	1·680	1·682	1·684	1·6838	1·6804	1·6714	1·6587	1·6449
0·5	1·671	1·681	1·686	1·687	1·687	1·6871	1·6817	1·6709	1·6580	1·6449
0·4	1·679	1·687	1·690	1·691	1·691	1·6896	1·6816	1·6698	1·6568	1·6449
0·3	1·687	1·693	1·694	1·693	1·692	1·6902	1·6804	1·6677	1·6550	1·6449
0·2	1·693	1·697	1·696	1·694	1·692	1·6898	1·6779	1·6646	1·6529	1·6449
0·1	1·698	1·699	1·697	1·693	1·690	1·6874	1·6738	1·6606	1·6504	1·6449
0·0	1·703	1·699	1·695	1·690	1·686	1·6827	1·6682	1·6558	1·6477	1·6449
−0·1	1·702	1·695	1·689	1·684	1·679	1·6756	1·6611	1·6503	1·6447	1·6449
−0·2	1·698	1·688	1·680	1·674	1·669	1·6657	1·6525	1·6442	1·6417	1·6449
−0·3	1·687	1·676	1·667	1·661	1·657	1·6535	1·6427	1·6378	1·6388	1·6449
−0·4	1·670	1·658	1·650	1·645	1·642	1·6391	1·6322	1·6314	1·6359	1·6449
−0·5	1·646	1·636	1·630	1·627	1·624	1·6231	1·6213	1·6252	1·6334	1·6449
−0·6	1·615	1·610	1·607	1·606	1·606	1·6066	1·6108	1·6195	1·6313	1·6449
−0·7	1·582	1·583	1·585	1·587	1·589	1·5911	1·6019	1·6150	1·6299	1·6449
−0·8	1·551	1·559	1·565	1·571	1·575	1·5792	1·5954	1·6122	1·6291	1·6449
−0·9	1·531	1·544	1·553	1·561	1·567	1·5722	1·5925	1·6116	1·6292	1·6449
−1·0	1·528	1·543	1·554	1·563	1·569	1·5744	1·5952	1·6141	1·6307	1·6449

$$\eta' = \frac{\delta}{\sqrt{(2f)}}\left(1+\frac{\delta^2}{2f}\right)^{-\frac{1}{2}}; \qquad t(f,\delta,\epsilon) = \frac{\delta+\lambda\left(1+\frac{\delta^2}{2f}-\frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1-\frac{\lambda^2}{2f}\right)}.$$

For note on accuracy of table see end of Section 9.

APPENDIX

On the calculation of Table IV

Before deciding on the method to be employed in calculating the tables a wide variety of methods were considered. We examined these with a view to finding one which would best combine the attributes of speediness, lack of opportunity for numerical errors, and adaptability to moderately large-scale work. In this appendix will be described the more promising of the methods tested, including the one finally adopted.

The non-central t being defined as in (1) we have for the probability that $t > t_0$, with the notation of Section 2,

$$P(f,\delta,t_0) = \frac{1}{2^{\frac{1}{2}f-1}\Gamma(\frac{1}{2}f)}\int_0^\infty \left\{\chi^{f-1}e^{-\frac{1}{2}\chi^2}\frac{1}{\sqrt{(2\pi)}}\int_{\frac{t_0\chi}{\sqrt{f}}}^\infty e^{-\frac{1}{2}(u-\delta)^2}du\right\}d\chi. \qquad \text{.....(41)}$$

We want to find pairs of values δ and t_0 such that $P(f,\delta,t_0)$ has specified values ϵ; i.e. we want to evaluate the quantities $t_0(f,\delta,\epsilon)$ and $\delta(f,t_0,\epsilon)$.

(i) *Direct Methods*

These are based on the idea of evaluating the right-hand side of (41) directly for various values of δ, f, and t_0 and then obtaining the quantities required by a process of inverse interpolation. An obvious way to evaluate $P(f, \delta, t_0)$ is by quadrature. For small even values of f, however, the following formulae:

$$P(0<t<\infty\,|\,f,\delta) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}(u-\delta)^2}\,du,$$

$$P(0<t<t_0\,|\,f,\delta) = \frac{1}{\sqrt{2\pi}}\exp\{-\tfrac{1}{2}\delta^2(1+f^{-1}t_0^2)^{-1}\}\sum_{r=0}^{\frac{1}{2}(f-2)}\frac{(2r)!}{(1+f^{-1}t_0^2)^r\,2^r\,r!}\,Hh_{2r}\{-\delta t_0(f+t_0^2)^{-\frac{1}{2}}\},$$

where
$$Hh_s(x) = \frac{1}{s!}\int_0^\infty u^s e^{-\frac{1}{2}(u+x)^2}\,du$$

were found to be more convenient to use. The Hh_s functions have been tabled by J. R. Airey (1931).

This modified procedure was still unduly lengthy and not well adapted to large scale work. It provides, however, a useful check on values obtained by other methods.

(ii) *Solution by means of a differential equation*

For fixed values of f and ϵ the equation $P(f, \delta, t_0) = \epsilon$ may be regarded as specifying δ and t_0 as implicit functions of each other. This implies that for constant ϵ,

$$\frac{dt_0}{d\delta} = -\left\{\frac{\partial P(f,\delta,t_0)}{\partial\delta}\Big/\frac{\partial P(f,\delta,t_0)}{\partial t_0}\right\}.$$

From (41) this is found to give

$$\frac{dt_0}{d\delta} = (f+t_0^2)^{\frac{1}{2}}\,J_f(-\delta t_0[f+t_0^2]^{-\frac{1}{2}}), \qquad \text{.....(42)}$$

where
$$J_f(x) = \frac{Hh_{f-1}(x)}{fHh_f(x)}.$$

It may be shown that
$$\frac{d}{dx}J_f(x) = fJ_f^2(x) - xJ_f(x) - 1. \qquad \text{.....(43)}$$

For given values of f and ϵ, the value of t_0 corresponding to $\delta = 0$ can be found in a straightforward manner since it is simply a probability level of Student's t. Starting from this initial value we can solve the differential equation (42) numerically and obtain the values of t_0 corresponding to non-zero values of δ. A trial of this method was made, taking $f = 6$, and $\epsilon = 0{\cdot}05$. The differential equation was solved by the Adams-Bashforth process. The values of higher derivatives of t with respect to δ at the point $\delta = 0$ are required in this process. By means of (43) the formulae for these quantities were obtained in quite a simple form.

This method gave promising results—it was suitable for large-scale work and there was comparatively little chance of inaccuracies entering the work—but it suffered from the defect that we could proceed to build up our table by small increments only of δ (actually increments of 0·05). As we wanted to be able to deal with large as well as small values of δ we decided that the process was not sufficiently speedy for our purposes.

It may be remarked, however, that a similar type of approach might be quite suitable in cases where the expression for the differential coefficient is simpler than was ours.

(iii) *Solution based on normal approximation*

The method finally adopted for calculating the greater part of the table is based on the use of Edgeworth's development of the normal probability function. In a recent paper E. A. Cornish & R. A. Fisher (1937) have given formulae, derived from this development, which make it possible to find percentage levels of a probability function when the cumulants of the function are given. These formulae are of most practical use when the probability function is itself nearly normal in form. With markedly non-normal distributions, the development may not converge, or at least more terms may be required than is practically convenient.

The cumulants of the non-central t-distribution are not difficult to find. The first three are given in equations (19), (20) and (21) and the values of $\sqrt{\beta_1}$ for different f and δ are given in Table I. It was pointed out in the discussion of Section 6 that the non-central t-distribution would often be markedly skew. It would appear, therefore, that it may not be a suitable distribution to express by a development of the normal function. Fortunately it is possible, as is pointed out in equation (26), to express problems demanding the quantities $t(f, \delta, \epsilon)$ and $\delta(f, t_0, \epsilon)$ in a form such that the solution can be based on a distribution much more nearly normal than that of non-central t.

It was shown that the statement

$$P(t > t_0 \,|\, f, \delta) = \epsilon$$

was equivalent to the statement

$$P(Y < \delta \,|\, f, t_0) = \epsilon,$$

where

$$Y = (-z)\sqrt{+}\, t_0 \chi f^{-\frac{1}{2}}.$$

(Here z is a unit normal deviate and χ is distributed in the χ-distribution with f degrees of freedom.) The problem of finding $\delta(f, t_0, \epsilon)$ is therefore equivalent to finding a percentage level of Y.

The cumulants of Y may be obtained immediately from the formulae

$$\kappa_r(Y) = \kappa_r(-z) + f^{-\frac{1}{2}r} t_0^r \kappa_r(\chi). \qquad \text{.....(44)}$$

In particular for $r = 2$, $\kappa_2(Y) = 1 + f^{-1} t_0^2 \kappa_2(\chi)$,

while for $r \neq 2$, $\kappa_r(Y) = f^{-\frac{1}{2}r} t_0^r \kappa_r(\chi)$.

Hence for all r the shape coefficients $\gamma_r(Y)$ increase from 0 to $\gamma_r(\chi)$ as t_0 increases from 0 to ∞. The γ's of Y are therefore always smaller than those of χ. But these latter are quite small even for small f. Hence the distribution of Y can never be far removed from normality. The distribution of Y is therefore suitable for development by Edgeworth's method and Cornish & Fisher's formulae may conveniently be used to provide percentage levels.

In this approach the problem of finding $\delta(f, t_0, \epsilon)$ appears as more fundamental than that of finding $t(f, \delta, \epsilon)$. Hence Table IV is designed to facilitate the direct calculation of $\delta(f, t_0, \epsilon)$. Its use to provide $t(f, \delta, \epsilon)$ demands a process of iteration which is not however difficult. The actual quantity $\lambda(f, t_0, \epsilon)$ which is tabled differs by a small amount from K_ϵ, the normal deviate which is exceeded with probability ϵ. This difference is given by the Cornish-Fisher formula. All the terms in this formula were used. This necessitated the use of the first six cumulants of χ. A method of obtaining these has already been described in a previous note (N. L. Johnson & B. L. Welch, 1939).

The greater part of the computation of Tables IV and V has been the work of one of us (N. L. Johnson). The remainder and much subsidiary checking work is due to Miss Catherine M. Thompson. We gratefully acknowledge her assistance.

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MISCELLANEA

(i) Stirling's formula with remainder*

By E. V. HUNTINGTON, Harvard University

We assume the following theorem as known:

$$n\,! = \sqrt{(2\pi)}\sqrt{n}\,n^n e^{-n}[e^P],$$

where $n = 2, 3, 4, \ldots$, and

$$\begin{aligned} P &= \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \frac{1}{1188n^9} - \frac{691}{360360n^{11}} + \ldots, \\ &= 0{\cdot}083,333n^{-1} - 0{\cdot}002,777n^{-3} + 0{\cdot}000,794n^{-5} - 0{\cdot}000,595n^{-7} \\ &\qquad + 0{\cdot}000,842n^{-9} - 0{\cdot}001,918n^{-11} + \ldots. \end{aligned}$$

We assume also that the error involved in stopping the series at any point is less (in absolute value) than the first term dropped. (For proof, see, for example, E. B. Wilson, *Advanced Calculus*, p. 456.)

Putting $Q = e^P$, and expanding and collecting terms, we find:

$$n\,! = \sqrt{(2\pi)}\sqrt{n}\,n^n e^{-n}[Q],$$

where
$$\begin{aligned} Q &= 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51{,}840n^3} - \frac{571}{2{,}488{,}320n^4} \\ &\quad + \frac{163{,}879}{209{,}018{,}880n^5} + \frac{5{,}246{,}819}{75{,}246{,}796{,}800n^6} - \frac{534{,}703{,}531}{902{,}961{,}561{,}600n^7} \\ &\quad - \frac{4{,}483{,}131{,}259}{86{,}684{,}309{,}913{,}600n^8} + \frac{432{,}261{,}921{,}612{,}371}{514{,}904{,}800{,}886{,}784{,}000n^9} + \ldots, \\ &= 1 + 0{\cdot}083{,}333{,}333n^{-1} + 0{\cdot}003{,}472{,}222n^{-2} \\ &\quad - 0{\cdot}002{,}681{,}327n^{-3} - 0{\cdot}000{,}229{,}472n^{-4} \\ &\quad + 0{\cdot}000{,}784{,}039n^{-5} + 0{\cdot}000{,}069{,}728n^{-6} - 0{\cdot}000{,}592{,}166n^{-7} \\ &\quad - 0{\cdot}000{,}051{,}718n^{-8} + 0{\cdot}000{,}839{,}499n^{-9} + \ldots. \end{aligned}$$

(For the terms up to and including n^{-7}, compare, for example, H. T. Davis, *Tables*, **1**, 180.)

Let $Q_2 =$ the sum of the Q-series up to and including the term in n^{-2}, $Q_3 =$ the sum of the Q-series up to and including the term in n^{-3}, etc.

Then we find:

$$\begin{aligned} Q &= 1 + R_1, && \text{where} && 0{\cdot}083,33n^{-1} < R_1 < 0{\cdot}085,07n^{-1}. \\ Q &= Q_1 + R_2, && \text{where} && 0{\cdot}002,07n^{-2} < R_2 < 0{\cdot}003,48n^{-2}. \\ Q &= Q_2 + R_3, && \text{where} && -0{\cdot}002,80n^{-3} < R_3 < -0{\cdot}002,59n^{-3}. \\ Q &= Q_3 + R_4, && \text{where} && -0{\cdot}000,23n^{-4} < R_4 < 0{\cdot}000,18n^{-4}. \\ Q &= Q_4 + R_5, && \text{where} && 0{\cdot}000,66n^{-5} < R_5 < 0{\cdot}000,82n^{-5}. \\ Q &= Q_5 + R_6, && \text{where} && -0{\cdot}000,24n^{-6} < R_6 < 0{\cdot}000,07n^{-6}. \\ Q &= Q_6 + R_7, && \text{where} && -0{\cdot}000,62n^{-7} < R_7 < -0{\cdot}000,39n^{-7}. \\ Q &= Q_7 + R_8, && \text{where} && -0{\cdot}000,06n^{-8} < R_8 < 0{\cdot}000,39n^{-8}. \\ Q &= Q_8 + R_9, && \text{where} && -0{\cdot}000,01n^{-9} < R_9 < 0{\cdot}000,88n^{-9}. \end{aligned}$$

* Presented to the American Mathematical Society, 8 April 1939.

(ii) A note on the interpretation of quasi-sufficiency

By M. S. BARTLETT

1. It has been shown by Fisher that if in problems of location we confine our attention to samples with the same configuration C, where C denotes the differences between the observations, then no statistical information in his sense is lost whatever reasonable statistic T we choose as an estimate of the unknown parameter θ. This result follows for probability laws of the type under consideration,

$$p(x \mid \theta) \equiv f(x-\theta)\,dx,$$

from the relation

$$p(S \mid \theta) = p(T \mid C, \theta)\,p(C), \qquad \text{.....(1)}$$

where S denotes the sample of observations. For the conditional statistic $T \mid C$—a random or statistical variable T varying for given or fixed C—I have when a relation like (1) exists used the term *quasi-sufficient*, in view of the correspondence of (1) with the relation for a sufficient statistic T,

$$p(S \mid \theta) = p(T \mid \theta)\,p(S \mid T). \qquad \text{.....(2)}$$

Conditional statistics also occur in the theory of more than one unknown, when the fixing of some statistical variables has the effect of eliminating unwanted parameters from our distributions. Confining our attention, however, to problems which are primarily problems in one unknown only, we require to examine relations of the type (1) further, in view of some recent comments by Welch* on the extent to which any conditional statistic like $T \mid C$ can claim to be sufficient.

From Fisher's theory of information, it follows that when estimating θ from two or more samples, we should naturally weight the different samples according to the information $I(C)$ in each. But Welch has pointed out that if we are concerned with interval estimation of θ from a single sample, or with the most efficient statistical test associated with a value θ_0, it does not appear that all our information, in the widest sense, is retained in $T \mid C$. If, for example, the only alternative to θ_0 happens to be θ_1, it is known that the appropriate criterion for discriminating θ_0 and θ_1 is $p(S \mid \theta_1)/p(S \mid \theta_0)$. While from equation (1) it follows that this criterion is equivalent to $p(T \mid C, \theta_1)/p(T \mid C, \theta_0)$, the former criterion must be referred to the distribution of S, not of $S \mid C$.

In partial answer to this criticism, it might be noticed that when considering $S \mid C$, we are not bound to choose the same significance level α for each C observed. Suppose we choose $\alpha(C_1) = \epsilon$ for the first sample. If the second sample gave a different configuration C_2, we might take $\alpha(C_2)$ such that the power of the test was a maximum for the two configurations C_1 and C_2 if we had chosen $\alpha(C_1)$ such that $\frac{1}{2}\{\alpha(C_1) + \alpha(C_2)\} = \epsilon$. Similarly for C_3, and so on. In the long run, the significance level adopted on this rule would be the average value $E_r\{\alpha_r(C_r)\}$, where $\alpha_r(C_r)$ denotes the significance level taken for the rth sample when making the rth test. For any finite number of samples the average significance level adopted is not exactly equal to ϵ, but the level becomes equal to ϵ in the limit; thus we could argue that, theoretically at least, we should eventually reach the most powerful test for a given significance level merely from consideration of $S \mid C$. In the above argument we allow the configuration distribution to be generated by the samples themselves. It is of course theoretical quibbling, for if we may assume that C has a definite distribution $p(C)$ which is already known, we should use this fact rather than wait for its confirmation by repeated sampling.

My own answer to the query of how far $T \mid C$ can be considered sufficient is that it is only sufficient provided that we do really agree to consider samples with the same configuration as that observed. The rejection altogether of the factor $p(C)$ has two consequences, (i) we have seen that a test might conceivably be derived on the basis of $p(C)$ more powerful than

* B. L. Welch. *Annals Math. Statist.* **10** (1939), 58–69.

the test based on $T \mid C$ with fixed significance level $\alpha(C) = \epsilon$, (iii) the fixing of C implies that whatever selection for C there might be distorting $p(C)$, the validity of tests based on $T \mid C$ will be unaffected.

The independence of the test from selection for C is the explicit advantage gained, though in addition it is possible that if our specification of the population is erroneous, that it is less so for $p(T \mid C)$ than $p(S)$; for example, if the configuration C is similar to the expected configuration on normal theory, it may prove more feasible to assume for a certain range of populations differing from normality that normal theory may still be approximately substituted for the unknown probability $p(T \mid C)$ than that we may do so for $p(S)$ or $p(T)$, though I have not investigated this point.

2. It is instructive to remember Fisher's comparison of the quasi-sufficient statistic $T \mid C$ with the sufficient statistic T, where T may be written in full, $T \mid n$, where n is the size of the sample. The equivalence of the two types of statistic when we regard C as fixed can be extended, in terms of our theoretical argument, if conversely we assume that n had in some problem a definite and known distribution from sample to sample. Consider similarly the test of significance of a regression coefficient. The orthodox theory is to consider the conditional statistic $b \mid \Sigma(x-\bar{x})^2$, where b is our estimate, and $\Sigma(x-\bar{x})^2$ the sum of squares of deviations of the independent variable x. Suppose for the sake of argument that the true variance of the residual dependent variable y_x was known to be unity, and the x's are such that $\Sigma(x-\bar{x})^2 = 1$ on Mondays and 1·44 on Tuesdays. Then for an 0·025 significance level (one tail), the usual practice would be to take 1·96 as the significance level for b (from $b_0 = 0$) on Mondays, and $1{\cdot}96/1{\cdot}2 = 1{\cdot}633$ on Tuesdays. The power of the test in relation to the alternative that $b_1 = 3{\cdot}92$ is 0·9860. But if we were satisfied with adjusting the significance level to be 0·025 merely in the long run for Mondays and Tuesdays together, we may raise the power of the test to its maximum value of 0·9878, by taking the Monday significance level at $b = 1{\cdot}87$ ($\alpha = 0{\cdot}0307$) and the Tuesday level at $b = 1{\cdot}723$ ($\alpha = 0{\cdot}0194$).

SUMMARY

In a theoretical note on the interpretation of quasi-sufficient statistics in problems of one unknown parameter, it is agreed with Welch that full sufficiency properties can only be claimed if we deliberately confine our attention to conditional samples of the type observed; but it is pointed out that the resulting inferences are independent of any selection operating on the variables that have been fixed.

(iii) The cumulants and moments of the binomial distribution, and the cumulants of χ^2 for a $(n \times 2)$-fold table

BY J. B. S. HALDANE, F.R.S.

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The first four cumulants of the distribution of χ^2 for a $(n \times 2)$-fold table when samples are finite, have already been given (Haldane, 1937). These and higher cumulants and moments can be calculated by a simpler method. Consider a sample of s, the probability of a success being p, and $p + q = 1$. Pearson (1919) pointed out that for moments of $(p+q)^s$ about its mean, the generating function is $(qe^{pt} + pe^{-qt})^s$, and Romanovsky (1923) gave a recurrence formula for the moments. That for the cumulants is much simpler.

Let $$U = qe^{pt} + pe^{-qt}.$$

Then the cumulant-generating function

$$K(t) \equiv \sum_{r=2}^{\infty} \frac{\kappa_r t^r}{r!} = s \log U.$$

To find the cumulants for $s = 1$, we note that

$$\frac{\partial}{\partial q} K(t) = \frac{e^{pt} - e^{-qt}}{U} - t,$$

$$\frac{\partial}{\partial t} K(t) = \frac{pq(e^{pt} - e^{-qt})}{U}$$

So
$$\frac{\partial}{\partial t} K(t) = pq \frac{\partial}{\partial q} K(t) + pqt,$$

or
$$\sum_{r=2}^{\infty} \frac{\kappa_r t^{r-1}}{(r-1)!} \equiv pq \sum_{r=2}^{\infty} \frac{t^r}{r!} \frac{d\kappa_r}{dq} + pqt.$$

Equating the coefficients of $t^r/r!$, we find

$$\kappa_2 = pq,$$

and if $r > 2$
$$\kappa_{r+1} = pq \frac{d\kappa_r}{dq}. \qquad \text{.....(1)}$$

Let $pq = c$, $p - q = g$; then

$$\kappa_{r+1} = c \frac{d\kappa_r}{dq}, \quad \frac{dc}{dq} = g, \quad \frac{dg}{dq} = -2, \quad g^2 = 1 - 4c.$$

Hence if $\kappa_{2r} = f(c)$,

$$\left.\begin{aligned} \kappa_{2r+1} &= gcf'(c), \\ \kappa_{2r+2} &= c(1-6c)f'(c) + c^2(1-4c)f''(c). \end{aligned}\right\} \qquad \text{.....(2)}$$

From these equations we can very rapidly calculate successive values of κ_r, since $\kappa_2 = c$, and find:

$$\kappa_1 = 0,$$
$$\kappa_2 = c,$$
$$\kappa_3 = cg,$$
$$\kappa_4 = c - 6c^2,$$
$$\kappa_5 = g(c - 12c^2),$$
$$\kappa_6 = c - 30c^2 + 120c^3,$$
$$\kappa_7 = g(c - 60c^2 + 360c^3),$$
$$\kappa_8 = c - 126c^2 + 1{,}680c^3 - 5{,}040c^4,$$
$$\kappa_9 = g(c - 252c^2 + 5{,}040c^3 - 20{,}160c^4),$$
$$\kappa_{10} = c - 510c^2 + 17{,}640c^3 - 151{,}200c^4 + 362{,}880c^5,$$
$$\kappa_{11} = g(c - 1{,}020c^2 + 52{,}920c^3 - 604{,}800c^4 + 1{,}814{,}400c^5),$$
$$\kappa_{12} = c - 2{,}046c^2 + 168{,}960c^3 - 3{,}160{,}080c^4 + 19{,}958{,}400c^5 - 39{,}916{,}800c^6. \qquad \text{.....(3)}$$

If each of the above cumulants be multiplied by s, the moments about the mean can now be calculated from the expressions given by Fisher (1928) and Haldane (1938). If $p = \frac{1}{2}$ we have

$$K(t) = s \log \cosh \tfrac{1}{2}t,$$

so
$$\kappa_2 = \frac{s}{4}, \quad \kappa_4 = -\frac{s}{8}, \quad \kappa_6 = \frac{s}{4}, \quad \kappa_8 = -\frac{17s}{16}, \quad \kappa_{10} = \frac{31s}{4}, \quad \kappa_{12} = -\frac{691s}{8},$$

while if q is very small we have for the cumulant-generating function of a Poisson series

$$K(t) = sc(e^t - 1 - t).$$

The coefficient of c^2 is $-[s^r+(-1)^r-3]$. So when q is small, but its square is not neglected, the first order correction to the Poisson cumulant-generating function is

$$K(t) = sq(e^t-1-t)+sq^2(e^{2t}-e^t).$$

The numerical coefficient of the highest power of c in κ_r is $(r-1)!$ when r is even, and $\frac{1}{2}(r-1)!$ when r is odd.

Consider a sample of s, in which a successes are recorded. Then

$$\chi^2 = \frac{(a-sp)^2}{spq}.$$

But $a-sp$ is the departure from the mean of the binomial distribution $(p+q)^s$. Hence the rth moment of the distribution of χ^2 (for one degree of freedom) about zero, is

$$\nu'_r = \frac{\mu_{2r}}{s^r c^r},$$

where μ_{2r} is the $2r$th moment of $(p+q)^s$.

But if μ'_r and κ'_r be the rth moment about the mean, and the rth cnmulant, of the χ^2 distribution, then

$$\mu_2 = \nu'_2-\nu'^2_1, \text{ etc.}, \quad \kappa'_2 = \mu'_2, \text{ etc.}$$

Making the necessary substitutions, we find, for the cumulants of χ^2 in terms of those of the binomial distribution:

$$\kappa'_1 = (sc)^{-1}\kappa_2,$$

$$\kappa'_2 = (sc)^{-2}(2\kappa_2^2+\kappa_4),$$

$$\kappa'_3 = (sc)^{-3}[8\kappa_2^3+2(5\kappa_3^2+6\kappa_2\kappa_4)+\kappa_6],$$

$$\kappa'_4 = (sc)^{-4}[48\kappa_2^4+48(5\kappa_2\kappa_3^2+3\kappa_2^2\kappa_4)+8(4\kappa_4^2+7\kappa_3\kappa_5+3\kappa_2\kappa_6)+\kappa_8],$$

$$\kappa'_5 = (sc)^{-5}[384\kappa_2^5+960(5\kappa_2^2\kappa_3^2+2\kappa_2^3\kappa_4)+80(25\kappa_3^2\kappa_4+16\kappa_2\kappa_4^2+28\kappa_2\kappa_3\kappa_5 + 6\kappa_2^2\kappa_6)+2(63\kappa_5^2+100\kappa_4\kappa_6+60\kappa_3\kappa_7+20\kappa_2\kappa_8)+\kappa_{10}],$$

$$\begin{aligned}\kappa'_6 = (sc)^{-6}[&3{,}840\kappa_2^6+9{,}600(10\kappa_2^3\kappa_3^2+3\kappa_2^4\kappa_4)+4{,}800(3\kappa_3^4+25\kappa_2\kappa_3^2\kappa_4\\ &+8\kappa_2^2\kappa_4^2+14\kappa_2^2\kappa_3\kappa_5+2\kappa_2^3\kappa_6)+40(132\kappa_4^3+672\kappa_3\kappa_4\kappa_5+189\kappa_2\kappa_5^2\\ &+226\kappa_3^2\kappa_6+300\kappa_2\kappa_4\kappa_6+180\kappa_2\kappa_3\kappa_7+30\kappa_2^2\kappa_8)+4(113\kappa_6^2+198\kappa_5\kappa_7\\ &+120\kappa_4\kappa_8+55\kappa_3\kappa_9+15\kappa_2\kappa_{10})+\kappa_{12}]. \qquad \text{.....(4).}\end{aligned}$$

We now substitute the values of κ_r given in equations (3) multiplied by s, putting

$$k = (pq)^{-1} = c^{-1}.$$

We therefore have, for the cumulants of χ^2 with one degree of freedom:

$$\kappa_1 = 1,$$

$$\kappa_2 = 2+(k-6)s^{-1},$$

$$\kappa_3 = 8+2(11k-56)s^{-1}+(k^2-30k+120)s^{-2},$$

$$\kappa_4 = 48+96(4k-19)s^{-1}+16(7k^2-125k+420)s^{-2}+(k^3-125k^2+1{,}680k-5{,}040)s^{-3},$$

$$\begin{aligned}\kappa_5 = 384&+960(7k-32)s^{-1}+400(15k^2-214k+648)s^{-2}+6(81k^3\\ &-3{,}908k^2+38{,}420k-98{,}496)s^{-3}+(k^4-510k^3+17{,}640k^2\\ &-151{,}200k+362{,}880)s^{-4},\end{aligned}$$

$$\begin{aligned}\kappa_6 = 3{,}840&+9{,}600(13k-58)s^{-1}+9{,}600(26k^2-327k+924)s^{-2}\\ &+40(1{,}729k^3-56{,}236k^2+459{,}024k-1{,}065{,}792)s^{-3}+4(501k^4\\ &-59{,}398k^3+1{,}289{,}244k^2-8{,}824{,}320k+18{,}555{,}840)s^{-4}\\ &+(k^5-2{,}046k^4+168{,}960k^3-3{,}160{,}080k^2+19{,}958{,}400k\\ &-39{,}916{,}800)s^{-5}. \qquad \text{.....(5)}\end{aligned}$$

When $p = \frac{1}{2}$, $k = 4$, and we have, for n degrees of freedom:

$$\kappa_1 = n,$$
$$\kappa_2 = 2ns^{-1}(s-1),$$
$$\kappa_3 = 8ns^{-2}(s-1)(s-2),$$
$$\kappa_4 = 16ns^{-3}(s-1)(3s^2-15s+17),$$
$$\kappa_5 = 128ns^{-4}(s-1)(s-2)(3s^2-21s+31).$$
$$\kappa_6 = 256ns^{-5}(s-1)(15s^4-210s^3+990s^2-1{,}950s+1{,}382). \quad \ldots\ldots(6)$$

If there are n samples, with different values of s, we have, for the cumulants of χ^2, where $h = \dfrac{1}{2pq}$, and $R_i = \Sigma s^{-i}$,

$$\kappa_1 = n,$$
$$\kappa_2 = 2n[1+(h-3)R_1],$$
$$\kappa_3 = 4n[2+(11h-28)R_1+(h^2-15h+30)R_2],$$
$$\kappa_4 = 8n[6+12(8h-19)R_1+4(14h^2-125h+210)R_2+(h^3-63h^2+420h-630)R_3],$$
$$\kappa_5 = 16n[24+120(7h-16)R_1+100(15h^2-107h+162)R_2+3(81h^3 - 1{,}954h^2+9{,}560h-12{,}312)R_3+(h^4-255h^3+4{,}410h^2-18{,}900h+22{,}680)R_4],$$
$$\kappa_6 = 32n[120+600(13h-29)R_1+600(52h^2-327h+462)R_2+10(1{,}729h^3 - 28{,}118h^2+114{,}756h-133{,}228)R_3+2(501h^4-29{,}699h^3+322{,}311h^2 - 1{,}103{,}040h+1{,}159{,}740)R_4+(h^5-1{,}023h^4+42{,}240h^3-395{,}010h^2 + 1{,}247{,}400h-1{,}247{,}400)R_5]. \quad \ldots\ldots(7)$$

When $p = q = \frac{1}{2}$, we have:

$$\kappa_1 = n,$$
$$\kappa_2 = 2(n-R_1),$$
$$\kappa_3 = 8(n-3R_1+4R_2),$$
$$\kappa_4 = 16(3n-18R_1+32R_2-17R_3),$$
$$\kappa_5 = 128(3n-30R_1+100R_2-135R_3+62R_4),$$
$$\kappa_6 = 256(15n-225R_1+1{,}200R_2-2{,}940R_3+3{,}332R_4-1{,}382R_5). \quad \ldots\ldots(8)$$

The first four of equations (5, 6, 7, 8) have already been given in a slightly different form by Haldane (1937). The limiting forms of equations (5) and (7) when s tends to infinity and k to zero, while $ks = g$, have been given by Haldane (1938). However, the expression for κ_6 there given is incorrect. The coefficient of R_1 in the expression for κ_6 should be 124,800.

The extension of equations (7) would be rather tedious. However, those of equations (6) and (8) would not be very difficult. The coefficient of $x^{2r}/2r!$ in the expansion of $\log\cosh t$ is the value of $(d/dx)^{2r-1}(1-\tanh^2 x)$ when $x = 0$, and can easily be calculated, since this differential coefficient is a polynomial in $\tanh x$. The equations for moments in terms of cumulants can easily be extended when all odd cumulants vanish. In this case a useful check can be obtained from the fact that when $s = 2$ the cumulant-generating function of χ^2 is $t + \log\cosh t$.

SUMMARY

Expressions are obtained for the first twelve cumulants of the binomial distribution, and a simple recurrence formula for further cumulants. The first six cumulants of χ^2 for a $(n \times 2)$-fold table when expectations are small, are deduced.

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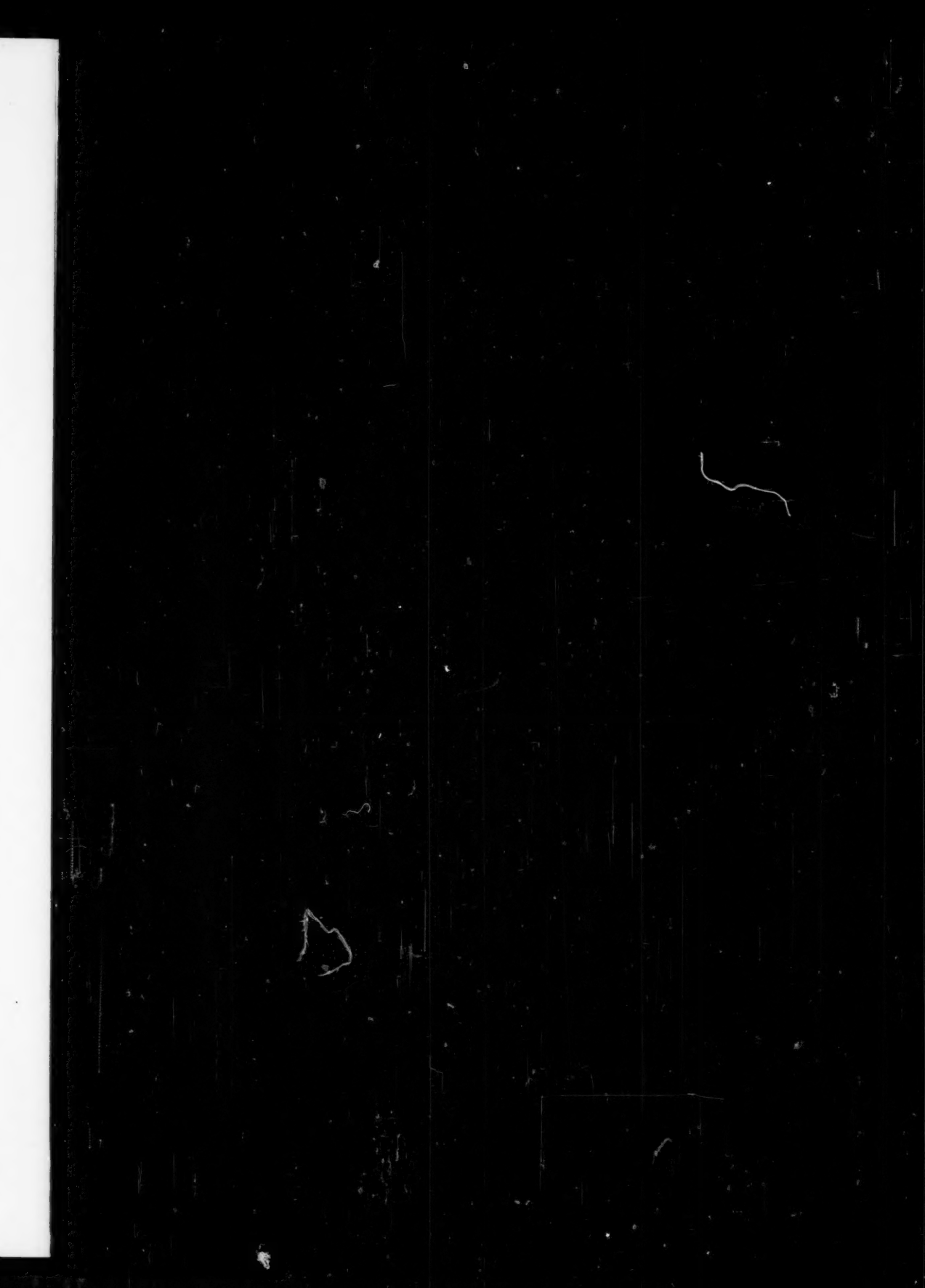
CAUSATION AND CORRELATION

(iv) The principles of the mathematical theory of correlation. By A. A. TSCHUPROW. Translated by M. KANTOROWITSCH. Wm. Hodge & Co. Ltd. Price 12*s*. 6*d*.

This book is a translation of an enlarged reproduction of lectures delivered by Professor Tschuprow at the University of Oslo, and originally printed in German about a decade ago. It will be of interest to many people in that it is a complete survey of correlation theory and its underlying principles. The theory is expounded along the now familiar classical lines which had their origin in Karl Pearson's writings, and although the application of this theory to practical problems is perhaps a little out of date, it still forms a necessary background which the student must acquire, and of which a proper understanding will always be essential.

The book as a whole shows an astonishing "patchiness" in writing. It may be the fault of the translator, but the fact remains that the meaning of whole paragraphs is sometimes very obscure, while at other times the ease and lucidity with which arguments are presented are unrivalled by any comparable treatise. This unevenness is unfortunate, since the book will be read more for the exposition of underlying principles than for its algebraic development of the theory; indeed it may be questioned whether those who have not previously mastered the elements of statistical theory and probability will obtain any profit from its reading. There is no clear discussion of probability and what it means, and the reader is left to find out from examples how a probability may be calculated. This cannot be deemed a fault, but seems to point to the book being useful only if previous knowledge of the subject is obtained elsewhere.

The chapter on stochastic connexion and functional relationship is good, and may be read with advantage by any statistical worker and teacher. The method of approach is the same as that of the present day, and will doubtless be followed for many years to come. It is, however, a little astonishing to find that the translator makes no use of the term "random variable", which has long passed into common use. The general discussion throughout the book and in particular in the chapter entitled "Object and value of Correlation Measurement" is stimulating, even if the reader is not always in agreement with the author. The mathematical development of the theory is set out with clarity and freshness, which make it enjoyable reading. The notation is a little cumbersome, but, once mastered, it does not prove difficult to follow.

Taken as a whole, this book is a worthy contribution to correlation literature, and it is surprising that no translation has been published until this date. It certainly should be read by all who attempt to gain an understanding of statistical theory.

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